Vector Calculus

— Additional Lecture Notes for 01002 Mathematics 1b

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Contents

Chapter 1

The real numbers

You only need to read this chapter if you want rigorous proof for important theorems about continuous functions. A crucial property of the real numbers is the nested interval theorem (Theorem 1.2).

In Appendix A we prove that the real numbers is the only ordered field (see below) that has this property and we also present a construction (or definition) of the real numbers and prove that the nested interval theorem is valid. You do not need the read the appendix and can just take the nested interval theorem as an axiom for the real numbers

1.1 Nested interval theorem

In [1] we had the definition of a field, i.e., we have a set $\mathbb{F}$ with an addition "+" and a multiplication "·" such that the associative, commutative, and distributive laws hold, there exists distinct neutral elements $0 \in \mathbb{F}$ and $1 \in \mathbb{F}$ for addition and multiplication, respectively, all elements have an additive inverse and all non zero elements have a multiplicative inverse.

The rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$ are all examples of fields and the only ones we will need. In $\mathbb{Q}$ and $\mathbb{R}$ we furthermore have an ordering "$\leq$". Given two numbers one of them is smaller than or equal to the other. This ordering makes $\mathbb{Q}$ and $\mathbb{R}$ into an ordered field:

Definition 1.1. A field $(\mathbb{F}, +, \cdot)$ is an *ordered fields* if it is equipped with a *total ordering* compatible with addition and multiplication, i.e., a relation "$\leq$" such that

$$\forall a \in \mathbb{F} : a \leq a \,, \qquad \text{reflexive} \qquad (1.1)$$

$$\forall a, b \in \mathbb{F} : a \leq b \wedge b \leq a \implies a = b \,, \qquad \text{antisymmetric} \qquad (1.2)$$

$$\forall a, b, c \in \mathbb{F} : a \leq b \wedge b \leq c \implies a \leq c \,, \qquad \text{transitive} \qquad (1.3)$$

$$\forall a, b \in \mathbb{F} : a \leq b \vee b \leq a \,, \qquad \text{total} \qquad (1.4)$$

compatible with addition and multiplication:

$$\forall a, b, c \in \mathbb{F} : a \leq b \implies a + c \leq b + c\,, \tag{1.5}$$

$$\forall a, b, c \in \mathbb{F} : a \leq b \wedge 0 \leq c \implies a \cdot c \leq b \cdot c\,. \tag{1.6}$$

Conditions (1.1), (1.2), and (1.3) is the the definition of an *ordering*. Together with condition (1.4) we have the definition of a *total ordering*. Finally, adding the two compatibility conditions (1.5) and (1.6) we have the definition of an *ordered field*.

Both $\mathbb{Q}$ and $\mathbb{R}$ with the usual definition of $\leq$ satisfies these axioms. But there are many more examples of ordered fields.

The following statement about the real numbers gives a precise meaning to the phrase that "there are no "holes" in the real axis".

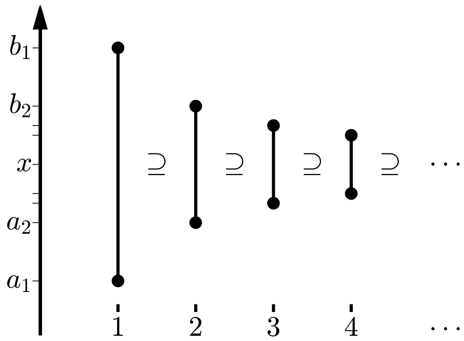

Figure 1.1: Nested intervals.

Theorem 1.2 (Nested interval theorem)**.** *If we have a nested sequence of closed intervals* $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \ldots$ *in* $\mathbb{R}$ *such that* $b_n - a_n \to 0$ *for* $n \to \infty$*. Then their intersection consists of a single number, i.e., there exists* $x \in \mathbb{R}$ *such that* $\bigcap_{n\in\mathbb{N}} [a_n, b_n] = \{x\}$*, see Figure 1.1.*

We will take this theorem as an axiom for the real numbers, but in Appendix A we give a construction of $\mathbb{R}$ and prove the theorem. We also show that it is the only ordered field with this property.

The precise meaning of "$b_n - a_n \to 0$ for $n \to \infty$" will be given in Definition 1.3 below.

1.2 Sequences

A *sequence* of real numbers is simply a map $F : \mathbb{N} \to \mathbb{R}$, but if let $x_n = F(n)$ then we will write $x_1, x_2, x_3, \ldots, x_n, \ldots$ or more compactly $(x_n)_{n\in\mathbb{N}}$.

If $(n_k)_{k\in\mathbb{N}}$ is strictly increasing sequence in $\mathbb{N}$, i.e., $n_k \in \mathbb{N}$ and $k > \ell \Rightarrow n_k > n_\ell$, then we call $(x_{n_k})_{k\in\mathbb{N}}$ a *subsequence* of $(x_n)_{n\in\mathbb{N}}$. If we think of the sequence as an infinite row of numbers then we simply remove some of the numbers.

Example 1.1. The sequence $(n^2)_{n\in\mathbb{N}}$ is a subsequence of $(n)_{n\in\mathbb{N}}$, $(n_k = k^2)$:

$$1, \cancel{2, 3}, 4, \cancel{5, 6, 7, 8}, 9, \cancel{10, 11, 12, 13, 14, 15}, 16, \cancel{17, \cdots, 24}, 25, \cdots.$$

If $k, \ell \in \mathbb{N}$ then $k, \ell > 0$ and multiplying with k and ℓ shows that $k \le \ell \Rightarrow k^2 \le k\ell \wedge k\ell \le \ell^2 \Rightarrow k^2 \le \ell^2$.

If x_n approaches a specific value as n gets bigger and bigger we say the sequence converges. More precisely

Definition 1.3. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$, i.e., a map $\mathbb{N} \to \mathbb{R} : n \mapsto x_n$. We say that it *converges* to $x \in \mathbb{R}$ and write $x_n \to x$ for $n \to \infty$ if

$$\forall \epsilon > 0\, \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n > n_0 \implies |x_n - x| < \epsilon\,. \tag{1.7}$$

The number x is called the *limit* of the sequence and we write $x = \lim_{n\to\infty} x_n$.

Example 1.2. The sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is convergent with limit 0. Indeed, given $\epsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\epsilon}$. For $n \in \mathbb{N}$ we now have that $n > n_0 \Rightarrow n > \frac{1}{\epsilon} \Rightarrow \frac{1}{n} < \epsilon$. For the first implication we used that the ordering is transitive and for the second we use that the ordering is compatible with multiplication (we multiply with $\frac{\epsilon}{n} > 0$ on both sides of the inequality).

If a sequence is convergent then the limit is unique:

Theorem 1.4. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $x, x' \in \mathbb{R}$, $x_n \to x$ for $n \to \infty$, and $x_n \to x'$, for $n \to \infty$ then $x = x'$.*

Proof. Given $\epsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow |x_n - x| < \epsilon/2$ and $n_1 \in \mathbb{N}$ such that $n > n_1 \Rightarrow |x_n - x'| < \epsilon/2$. If $n > \max\{n_0, n_1\}$ then we have $|x - x'| = |x - x_n + x_n - x'| \le |x - x_n| + |x_n - x'| < \epsilon/2 + \epsilon/2 = \epsilon$. As ϵ was arbitrary we must have $x = x'$. □

If a sequence does not converges then it is called *divergent*. If a sequence grows or decreases without bounds it is divergent, but we say that x_n tends to infinity and write $x_n \to \infty$ for $n \to \infty$ or $\lim_{n\to\infty} x_n = \infty$ if

$$\forall c \in \mathbb{R}\, \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n > n_0 \implies x_n > c\,. \tag{1.8}$$

Likewise, we say x_n tends to minus infinity and we write $x_n \to -\infty$ for $n \to \infty$ or $\lim_{n\to\infty} x_n = -\infty$ if

$$\forall c \in \mathbb{R}\, \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n > n_0 \implies x_n < c\,. \tag{1.9}$$

Example 1.3. The sequence $(n)_{n\in\mathbb{N}}$ is divergent, but tends to infinity. Indeed, given $C \in \mathbb{R}$ we can find $n_0 \in \mathbb{N}$ such that $n_0 > C$. Let $n \in \mathbb{N}$, if $n > n_0$ then $n > C$ (the transitive rule).

A subsequence of a convergent sequence is convergent with the same limit:

Lemma 1.5. *If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence. Then a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is convergent too with the same limit, i.e., $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$.*

Proof. If $x = \lim_{n\to\infty} x_n \in \mathbb{R}$ and $\epsilon > 0$ then we can find n_0 such that $n > n_0 \Rightarrow |x_n - x| < \epsilon$. As $(n_k)_{k\in\mathbb{N}}$ is increasing we have $k > n_0 \Rightarrow n_k > n_0 \Rightarrow |x_{n_k} - x| < \epsilon$. □

Example 1.4. As the sequence $(\frac{1}{n^2})_{n\in\mathbb{N}}$ is a subsequence of $(\frac{1}{n})_{n\in N}$ and the latter is convergent with limit 0 we have $\frac{1}{n^2} \to 0$ for $n \to \infty$.

The same is true if $(x_n)_{n\in\mathbb{N}}$ tends to plus or minus infinity:

Lemma 1.6. *If $\lim_{k\to\infty}(x_n)_{n\in\mathbb{N}}$ tend to plus or minus infinity. Then a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ does too.*

Proof. If $\lim_{n\to\infty} = \infty$ and $C \in \mathbb{R}$ is given then we can find n_0 such that $n > n_0 \Rightarrow x_n > C$. Again we have $k > n_0 \Rightarrow n_k > n_0 \Rightarrow x_{n_k} > C$. The case $\lim_{n\to\infty} = -\infty$ is similar. □

Example 1.5. As the sequence $(n^2)_{n\in\mathbb{N}}$ is a subsequence of $(n)_{n\in N}$ and the latter tends to infinity we have $n^2 \to \infty$ for $n \to \infty$.

Limits preserve inequalities:

Lemma 1.7. *If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, $c \in \mathbb{R}$, and $x_n \le c$ for all $n \in \mathbb{N}$ then $\lim_{n\to\infty} x_n \le c$.*

Proof. Assume the opposite: $\lim_{n\to\infty} x_n > c$. Then we can find $\epsilon > 0$ such that $c + \epsilon < \lim_{n\to\infty} x_n$. We can now find $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow x_n > c + \epsilon > c$, but that contradicts $x_n \le c$. □

Remark 1.8. Limits do *not* preserves strict inequalities. Indeed, if $x_n = \frac{1}{n}$ then $x_n > 0$, but $\lim_{n\to\infty} x_n = 0$.

If $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are two sequences then we can add them and form a new sequence $(x_n + y_n)_{n\in\mathbb{N}}$. we can also multiply the sequence $(x_n)_{n\in\mathbb{N}}$ by a number $c \in \mathbb{R}$ and form the sequence $(cx_n)_{n\in\mathbb{N}}$. These two operations turn the space of sequences into a vector space.

We can also multiply two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ and form a new sequence $(x_n y_n)_{n\in\mathbb{N}}$. If $x_n \neq 0$ for all $n \in \mathbb{N}$ then we can form the new sequence $\left(\frac{1}{x_n}\right)_{n\in\mathbb{N}}$. All these operation preserves convergence:

Theorem 1.9. *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two convergent sequences. Then $(x_n + y_n)_{n\in\mathbb{N}}$ and $(x_n y_n)_{n\in\mathbb{N}}$ are convergent with limits*

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty}(x_n) + \lim_{n\to\infty}(y_n)\,, \tag{1.10}$$

$$\lim_{n\to\infty}(x_n y_n) = \lim_{n\to\infty}(x_n)\lim_{n\to\infty}(y_n)\,. \tag{1.11}$$

If $x_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}(x_n) \neq 0$. Then $\left(\frac{1}{x_n}\right)_{n\in\mathbb{N}}$ is convergent and

$$\lim_{n\to\infty}\left(\frac{1}{x_n}\right)_{n\in\mathbb{N}} = \frac{1}{\lim_{n\to\infty}(x_n)}. \tag{1.12}$$

Proof. Let $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$, and let $\epsilon > 0$ be given .
(1.10): As $x_n \to x$ we can find $n_1 \in \mathbb{N}$ such that $n > n_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2}$. Similar we can find $n_2 \in \mathbb{N}$ such that $n > n_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2}$. If we put $n_0 = \max\{n_1, n_2\}$ and $n > n_0$ then

$$|x_n + y_n - x - y| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(1.11): As $x_n \to x$ we can find $n_1 \in \mathbb{N}$ such that $n > n_1 \Rightarrow |x_n - x| < \frac{\epsilon}{2|y|+2}$. Similar we can find $n_2 \in \mathbb{N}$ such that $n > n_2 \Rightarrow |y_n - y| < \frac{\epsilon}{2|x|+2}$ and we can find $n_3 \in \mathbb{N}$ such that $n > n_3 \Rightarrow |y_n| < |y| + 1$. If we put $n_0 = \max\{n_1, n_2, n_3\}$ and $n > n_0$ then

$$\begin{aligned}|x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \leq |x_n y_n - x y_n| + |x y_n - xy| \\ &= |x_n - x||y_n| + |x||y_n - y| < \frac{\epsilon}{2|y|+2}(|y|+1) + \frac{\epsilon}{2|x|+2}|x| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

(1.12): As $x_n \to x$ and $x \neq 0$ we can find $n_1 \in \mathbb{N}$ such that $n > n_1 \Rightarrow |x_n - x| < \frac{\epsilon|x^2|}{2}$ and we can find $n_2 \in \mathbb{N}$ such that $n > n_2 \Rightarrow |x_n| > \frac{|x|}{2}$. If we put $n_0 = \max\{n_1, n_2\}$ and $n > n_0$ then

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \frac{|x - x_n|}{|x x_n|} < \frac{|x - x_n|}{|x\frac{x}{2}|} = \frac{2|x - x_n|}{|x^2|} < \frac{2}{|x^2|}\frac{\epsilon|x^2|}{2} = \epsilon. \qquad \square$$

Corollary 1.10. *If $x_n \to x \in \mathbb{R}$ for $n \to \infty$ and $c \in \mathbb{R}$ then $cx_n \to cx$ for $n \to \infty$.*

Proof. Letting $y_n = c$ this follows from Theorem 1.9 (1.11). $\square$

Remark 1.11. This shows that convergent sequences is a subspace of the vector space of sequences. Furthermore, mapping a convergent sequence to its limit is a linear map.

Example 1.6. As $\frac{1}{n} \to 0$ for $n \to \infty$ and $\frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n}$ this gives us a new proof that $\frac{1}{n^2} \to 0$ for $n \to \infty$. Using induction it is not hard to show that for any $k \in \mathbb{N}$ we have $\frac{1}{n^k} \to 0$ for $n \to \infty$.

Corollary 1.12. *If $x_n \leq y_n$ for all $n \in \mathbb{N}$, $x_n \to x \in \mathbb{R}$ and $y_n \to x \in \mathbb{R}$ for $n \to \infty$ then $x \leq y$.*

Proof. We have $x_n - y_n \leq 0$ so by Lemma 1.7 $x - y \leq 0$, i.e, $x \leq y$. $\square$

The nested interval theorem implies that a bounded sequence has a convergent subsequence.

Theorem 1.13. *A bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$, i.e., we have $a_1, b_1 \in \mathbb{R}$ such that $x_n \in [a_1, b_1]$ for all $n \in \mathbb{N}$. We now look at the midpoint $c = \frac{a+b}{2}$. If there are infinitely many elements of the sequence in the interval $[a_1, c]$ then we put $a_2 = a_1$ and $b_2 = c$. Otherwise we must have infinitely many elements of the sequence in the interval $[c, b_1]$ and then we put $a_2 = c$ and $b_2 = b_1$. Continuing this way, halving the intervals and picking a half with infinitely many elements from the sequence, we obtain for each $n \in \mathbb{N}$ an interval $[a_n, b_n]$ with infinitely many elements from the sequence and with length $b_n - a_n = \frac{b-a}{2^n} \to 0$ for $n \to \infty$. Theorem 1.2 now tells us that there is an $x \in \mathbb{R}$ such that $\bigcap_{n\in\mathbb{N}}[a_n, b_n] = \{x\}$.

We now let $n_1 = 1$ and recursively pick $n_{k+1} > n_k$ such that $x_{n_k} \in [a_k, b_k]$: Clearly $x_{n_1} = x_1 \in [a_1, b_1]$. Now suppose we have $1 = n_1 < n_2 < \cdots < n_k$ such that $x_{n_k} \in [a_k, b_k]$ and consider the interval $[a_{k+1}, b_{k+1}]$ we have infinitely many elements from the sequence in this interval. So $\{n \in \mathbb{N} \mid x_n \in [a_{k+1}, b_{k+1}]\} \setminus \{1, 2, \ldots, n_k\}$ is non empty and hence contain an element n_{k+1} we clearly have $n_{k+1} > n_k$ and $x_{n_{k+1}} \in [a_{k=1}, b_{k+1}]$.

We have $x, x_{n_k} \in [a_k, b_k]$ so $|x_{n_k} - x| \leq b_k - a_k \to 0$ for $k \to \infty$. Thus $x_{n_k} \to x$ for $k \to \infty$ and we have found a convergent subsequence. □

A sequence can have many convergent subsequences with different limits.

Example 1.7. Consider the sequence in $\mathbb{Z} \times \mathbb{Z}$ depicted in Figure A.1 left. Let $(p_n, q_n)_{n\in\mathbb{N}}$ be the subsequence where the points with $q = 0$ are skipped, i.e., we have $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) = \{(p_n, q_n) \mid n \in \mathbb{N}\}$. Now consider the sequence $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$ in $\mathbb{Q}$. We have all possible numerators and denominators so we have $\mathbb{Q} = \{\frac{p_n}{q_n} \mid n \in \mathbb{N}\}$. Furthermore, as $\frac{p_n}{q_n} = \frac{kp_n}{kq_n}$ all rational numbers appears infinitely many times in the sequence. So for any rational number $\frac{p}{q} \in \mathbb{Q}$ we can find a subsequence $(\frac{p_{n_k}}{q_{n_k}})_{k\in\mathbb{N}}$ such that $\frac{p_{n_k}}{q_{n_k}} = \frac{p}{q}$ for all $k \in \mathbb{N}$. We have $\frac{p_{n_k}}{q_{n_k}} \to \frac{p}{q}$ for $k \to \infty$. In other words, any rational number is the limit of a (constant) subsequence of $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$. In Exercise 1.6 you will be asked to show that any real number is the limit of a subsequence of $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$.

1.3 Supremum and infimum

The intervals $[a, b]$ and $]a, b]$ have a maximal element namely b. In contrast the intervals $[a, b[$ and $]a, b[$ do not have a maximal element, still the number b seems to play a similar role. Likewise the intervals $[a, b]$ and $[a, b[$ have a minimal element namely a while the intervals $]a, b[$ and $]a, b]$ do not have a minimal element, here the number a seems to play a similar role. If we have an arbitrary subset $A \subseteq \mathbb{R}$ the situation is perhaps not so obvious.

Definition 1.14. Let $A \subseteq \mathbb{R}$. If there exist a number $c \in \mathbb{R}$ such that $x \leq c$ for all $x \in A$ then we say that A is *bounded from above* and we call c an *upper bound*.

If there exist a number $c \in \mathbb{R}$ such that $x \geq c$ for all $x \in A$ then we say that A is *bounded from below* and we call c a *lower bound*.

If c is an upper bound for a set A then all larger numbers are also upper bounds for A. So we see that the set of upper bounds is an interval. It follows from Theorem 1.2 that the set of upper bounds has a minimal element:

Theorem 1.15. *If $A \subseteq \mathbb{R}$ is non empty and bounded from above then there exists a least upper bound called the* supremum*, i.e., the set of upper bounds has a minimum. We write*

$$\sup A = \min\{c \in \mathbb{R} \mid \forall x \in A : x \leq c\}\,. \tag{1.13}$$

Similar, if A is bounded from below then there is a largest lower bound called the infimum*. We write*

$$\inf A = \max\{c \in \mathbb{R} \mid \forall x \in A : x \geq c\}\,. \tag{1.14}$$

Proof. Let $A^{\text{upp}} = \{c \in \mathbb{R} \mid \forall x \in A : x \leq c\}$ be the set of upper bounds. Both A and A^{upp} are assumed to be non empty so we can pick $a \in A$ and $b \in A^{\text{upp}}$. If $a = b$ then $\max A = a = b = \min A^{\text{upp}}$ and we are done. Otherwise we look at the midpoint $\frac{a_1+b_1}{2}$. If it is an upper bound for A then we put $a_1 = a$ and $b_1 = \frac{a+b}{2}$ otherwise we pick $a_1 \in A$ such that $a_1 \geq \frac{a+b}{2}$ and put $b_1 = b$. We now have $a_1 \in A$, $b_1 \in A^{\text{upp}}$, and $b_1 - a_1 \leq \frac{b-a}{2}$. Continuing this way we either arrive at numbers $a_n \in A$ and $b_n \in A^{\text{upp}}$, where $a_n = b_n$ and we are done. Otherwise we obtain sequences $a_a \leq a_2 \leq \cdots \leq a_n \leq \ldots$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq \ldots$, where $b_n - a_n \leq \frac{b-a}{2^n} \to 0$ for $n \to \infty$. By Theorem 1.2 we have $\bigcap_{n\in\mathbb{N}}[a_n, b_n] = \{x_0\}$ for some $x_0 \in \mathbb{R}$. We have $b_n \to x_0$ for $n \to \infty$ so x_0 is an upper bound for A. Conversely, $a_n \to x_0$ for $n \to \infty$ so there are no smaller upper bound. Thus $x_0 = \min A^{\text{upp}}$.

The case of the infimum can be proved in the same manner. Or we can note that $\inf A = -\sup(-A) = -\sup\{-x \mid x \in A\}$. □

Remark 1.16. If there no upper bound then we write $\sup A = \infty$ and if there is no lower bound we write $\inf A = -\infty$.

Remark 1.17. If $A \neq \emptyset$ then $\inf A \leq \sup A$. But the set of lower or upper bounds for $\emptyset$ is $\mathbb{R}$ so $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

Example 1.8.

$$\begin{aligned} \inf]a,b[= \inf[a,b] &= a\,, & \sup]a,b[= \sup[a,b] &= b\,, \\ \inf \mathbb{N} &= 1\,, & \sup \mathbb{N} &= \infty\,, \\ \inf \mathbb{Z} &= -\infty\,, & \sup \mathbb{Z} &= \infty\,, \\ \inf\left\{\tfrac{1}{n}\middle| x > 1\right\} &= 0\,, & \sup\left\{\tfrac{1}{x}\middle| x > 1\right\} &= 1\,, \end{aligned}$$

If $f : A \to \mathbb{R}$ is a function then we often use the notation

$$\begin{aligned} \sup f &= \sup_{x\in A} f(x) = \sup\{f(x) \mid x \in A\}\,, \\ \inf f &= \inf_{x\in A} f(x) = \inf\{f(x) \mid x \in A\}\,. \end{aligned} \tag{1.15}$$

Example 1.9. $\inf_{x>1} \frac{1}{x} = 0$ and $\sup_{x>1} \frac{1}{x} = 1$

In particular for a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ we have the notation

$$\begin{aligned} \sup\,(x_n)_{n\in\mathbb{N}} &= \sup_{n\in\mathbb{N}} x_n = \sup\{x_n \mid n\in\mathbb{N}\}\,, \\ \inf\,(x_n)_{n\in\mathbb{N}} &= \inf_{n\in\mathbb{N}} x_n = \inf\{x_n \mid n\in\mathbb{N}\}\,. \end{aligned} \tag{1.16}$$

Example 1.10. $\inf_{n\in\mathbb{N}} \frac{1}{n} = 0$ and $\sup_{n\in\mathbb{N}} \frac{1}{n} = 1$

Observe that if $A \subseteq B \subseteq \mathbb{R}$ then an upper bound for B is also an upper bound for A. So $\sup B$ is an upper bound for A and hence we must have $\sup A \leq \sup B$. Likewise $\inf A \geq \inf B$.

1.3.1 Limes inferior and limes superior

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ then we have

$$\{x_n \mid n\in\mathbb{N}\} \supseteq \{x_n \mid n\geq 2\} \supseteq \cdots \supseteq \{x_n \mid n \geq k\} \supseteq \cdots .$$

Hence

$$\sup_{n\in\mathbb{N}} x_n \geq \sup_{n\geq 2} x_n \geq \cdots \geq \sup_{n\geq k} x_n \geq \sup_{n\geq k+1} x_n \geq \cdots .$$

and

$$\inf_{n\in\mathbb{N}} x_n \leq \inf_{n\geq 2} x_n \leq \cdots \leq \inf_{n\geq k} x_n \leq \inf_{n\geq k+1} x_n \leq \cdots .$$

This gives rise to the following definition

Definition 1.18. Let $(x_n)_{n\in\mathbb{N}}$ be sequence in $\mathbb{R}$. Limes inferior and limes superior is defined by

$$\liminf x_n = \lim_{k\to\infty} \inf_{n\geq k} x_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n\,, \tag{1.17}$$

$$\limsup x_n = \lim_{k\to\infty} \sup_{n\geq k} x_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} x_n\,. \tag{1.18}$$

Remark 1.19. Observe that we have $\liminf x_n \leq \limsup x_n$ and

$$\inf_{n\geq k} x_n \underset{k\to\infty}{\nearrow} \liminf x_n\,, \qquad \sup_{n\geq k} x_n \underset{k\to\infty}{\searrow} \limsup x_n\,.$$

Example 1.11.

$$\begin{aligned} \liminf n &= \infty\,, & \limsup n &= \infty\,, \\ \liminf \frac{1}{n} &= 0\,, & \limsup \frac{1}{n} &= 0\,, \\ \liminf (-1)^n + \frac{1}{n} &= -1\,, & \limsup (-1)^n + \frac{1}{n} &= 1\,, \end{aligned}$$

Lemma 1.20. *A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is bounded from below if and only if $\liminf x_n > -\infty$. It is bounded from above if and only if $\limsup x_n < \infty$.*

Proof. If we have a $c \in \mathbb{R}$ such that $x_n \geq c$ for all $n \in \mathbb{N}$ then $\inf_{n\geq k} x_n \geq c$ for all $k \in \mathbb{N}$ and hence $\liminf x_n \geq c > -\infty$. Conversely, if $\liminf x_n > -\infty$ then there exist a $k \in \mathbb{N}$ such that $\inf_{n\geq k} x_n > \liminf x_n - 1$. If we put $c = \min\{x_1, x_2, \ldots, x_{k-1}, \inf_{n\geq k} x_n - 1\}$ then $x_n \geq c$ for all $n \in \mathbb{N}$, i.e., the sequence is bounded from below. The statement about limes superior is proved similarly. □

Lemma 1.21. *A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is convergent with limit $x \in \mathbb{R}$ if and only if $\liminf x_n = \limsup x_n = x$.*

Furthermore, $x_n \to \infty$ for $n \to \infty$ if and only if $\liminf x_n = \infty$ and $x_n \to -\infty$ for $n \to \infty$ if and only if $\limsup x_n = -\infty$.

Proof. Suppose $x_n \to x \in \mathbb{R}$ for $n \to \infty$. If $\epsilon > 0$ then we can find n_0 such that $|x_n - x| < \epsilon$ for $n > n_0$. Then $x_n \in [x-\epsilon, x+\epsilon]$ for $n > n_0$, hence $\inf_{n>n_0} x_n, \sup_{n>n_0} x_n \in [x-\epsilon, x+\epsilon]$ and $\liminf x_n, \limsup x_n \in [x-\epsilon, x+\epsilon]$. As ϵ was arbitrary we must have $\liminf x_n = \limsup x_n = x$. Conversely, suppose $\liminf x_n = \limsup x_n = x \in \mathbb{R}$ and we are given $\epsilon > 0$. We can find $n_1 \in \mathbb{N}$ such that $\inf_{n>n_1} x_n > x - \epsilon$ and $n_2 \in \mathbb{N}$ such that $\sup_{n>n_2} x_n < x + \epsilon$. If $n_0 = \max\{n_1, n_2\}$ then $n > n_0 \Rightarrow x_n \in]x-\epsilon, x+\epsilon[$.

If $x_n \to \infty$ for $n \to \infty$ and $c \in \mathbb{R}$ then we can find n_0 such that $x > c$ for $n > n_0$. Then $\inf_{n>n_0} x_n \geq c$ and hence $\liminf x_n \geq c$. As c was arbitrary we must have $\liminf x_n = \infty$. Conversely, if $\liminf x_n = \infty$ and we are given $c \in \mathbb{R}$ then we can find n_0 such that $\inf_{n>n_0} x_n \geq c$ and hence $x_n \geq c$ for all $n > n_0$. The proof of the last statement is similar. □

Lemma 1.22. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Then there exists a subsequence x_{n_k} such that $x_{n_k} \to \liminf x_n$ for $k \to \infty$. Likewise there exists a subsequence x_{n_k} such that $x_{n_k} \to \limsup x_n$ for $k \to \infty$.*

Proof. If $\liminf x_n \in \mathbb{R}$ we let $a_k = \inf_{n>k} x_n$. Then $a_k > -\infty$ and $a_k \to \liminf x_n$ for $k \to \infty$. Choose $n_1 > 1$ such that $|x_{n_1} - a_1| < 1$. Choose $n_2 > n_1$ such that $|x_{n_2} - a_{n_1}| < \frac{1}{2}$. Choose $n_3 > n_2$ such that $|x_{n_3} - a_{n_2}| < \frac{1}{3}$ and so on. That is, we recursively choose $n_{k+1} > n_k$ such that $|x_{n_{k+1}} - a_{n_k}| < \frac{1}{k+1}$. Then we have that

$$\begin{aligned}
|x_{n_k} - \liminf x_n| &= |x_{n_k} - a_{n_{k-1}} + a_{n_{k-1}} - \liminf x_n| \\
&\leq |x_{n_k} - a_{n_{k-1}}| + |a_{n_{k-1}} - \liminf x_n| \\
&< \frac{1}{k} + |a_{n_{k-1}} - \liminf x_n| \to 0 + 0 = 0\,, \quad \text{for } k \to \infty\,.
\end{aligned}$$

If $\liminf x_n = -\infty$ then $(x_n)_{n\in\mathbb{N}}$ is not bounded from below. So we can choose $n_1 \in \mathbb{N}$ such that $x_{n_1} < -1$, can choose $n_2 > n_1$ such that $x_{n_2} < -2$ and so on. That is, we recursively choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} < -k - 1$. Now $x_{n_k} \to -\infty$ for $k \to \infty$.

If $\liminf x_n = \infty$ the statement is Lemma 1.21. The statement about limes superior is proved similarly. □

1.4 Open and closed sets

Definition 1.23. A set $U \subseteq \mathbb{R}$ is called *open* if we for all $x \in U$ can find a $r > 0$ such that $]x - r, x + r[\subseteq U$.

So a subset U is open if any point in U can move a little bit to the left and right and still be in U. We see that an open interval $]a, b[$ is open, that $\mathbb{R}$ is open and that $\emptyset$ is open.

Arbitrary unions of open sets are open:

Theorem 1.24. *If $(U_j)_{j \in J}$ is an arbitrary collection of open sets in $\mathbb{R}$ then their union $\bigcup_{j \in J} U_j$ is open. (J is some index set).*

Proof. Let $x \in \bigcup_{j \in J} U_j$ then $x \in U_{j_0}$ for $j_0 \in J$. As U_{j_0} is open we can find $r > 0$ such that $]x - r, x + r[\subseteq U_{j_0}$. But $U_{j_0} \subseteq \bigcup_{j \in J} U_j$ so $]x - r, x + r[\subseteq U$. $\square$

Finite intersections of open sets are open:

Theorem 1.25. *If $U_1, U_2, \ldots, U_n \subseteq \mathbb{R}$ are open sets then their intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is open.*

Proof. Let $x \in U_1 \cap \cdots \cap U_n$. For $k = 1, \ldots, n$ the set U_k is open so we can find r_k such that $]x - r_k, x + r_k[\subseteq U_k$. Put $r = \min\{r_1, \ldots, r_n\}$. For $k = 1, \ldots, n$ we then have $]x - r, x + r[\subseteq]x - r_k, x + r_k[\subseteq U_k$. But then $]x - r, x + r[\subseteq U_1 \cap \cdots \cap U_n$. $\square$

Definition 1.26. Let $A \subseteq \mathbb{R}$ be an arbitrary subset. A subset $U \subseteq A$ is called *open relative to* A if there exist an open set $U' \subseteq \mathbb{R}$ such that $U = A \cap U'$.

Lemma 1.27. *The following is equivalent for a subset $U \subseteq A$:*

1. *U is open relative to A.*
2. *For all $x \in U$ have an $r > 0$ such that $A \cap]x - r, x + r[\subseteq U$.*

Proof. $1 \Rightarrow 2$: If $U \subseteq A$ is open relative to A then we have an open set $U' \subseteq \mathbb{R}$ such that $U = A \cap U'$. If $x \in U$ then we can find $r > 0$ such that $]x - r, x + r[\subseteq U'$, but then $A \cap]x - r, x + r[\subseteq S \cap U = U'$.
$2 \Rightarrow 1$: For each $x \in U$ we choose $r_x > 0$ such that $A \cap]x - r_x, x + r_x[\subseteq U$. We now put $U' = \bigcup_{x \in U}]x - r_x, x + r_x[$. Then U' is open and $U \subseteq A \cap U'$. On the other hand $A \cap U' = \bigcup_{x \in U'} (A \cap]x - r_x, x + r_x[) \subseteq U$. Hence $U = A \cap U'$. $\square$

Theorem 1.28. *Let $A \subseteq \mathbb{R}$ be an arbitrary subset. We have the following properties of relative open sets:*

1. *If $(U_j)_{j \in J}$ is an arbitrary collection of relative open sets in A then their union $\bigcup_{j \in J} U_j$ is relative open.*
2. *If $U_1, U_2, \ldots, U_n \subseteq A$ are relative open sets in A then $U_1 \cap U_2 \cap \cdots \cap U_n$ is relative open.*

Proof. Suppose $(U_j)_{j\in J}$ are relative open sets. For each $j \in J$ we can find an open set $U_j \subseteq \mathbb{R}$ such that $U_j = A \cap U_j'$. If we put $U' = \bigcup_{j\in J} U_j'$ then U' is open and $A \cap U' = \bigcup_{j\in J}(A \cap U_j') = \bigcup_{j\in J} U_j$ so $\bigcup_{j\in J} U_j$ is relative open.

Now suppose $U_1, \ldots, U_n \subseteq A$ are relative open sets. for each $k = 1, \ldots, n$ we can find an open set $U_k' \subseteq \mathbb{R}$ such that $U_k = A \cap U_k'$. If we put $U' = U_1' \cap \cdots \cap U_n'$ then U' is open and

$$A \cap U' = A \cap U_1' \cap \cdots \cap U_n' = (A \cap U_1') \cap \cdots \cap (A \cap U_n') = U_1 \cap \cdots \cap U_n \,. \qquad \square$$

Definition 1.29. Let $A \subseteq \mathbb{R}$ be arbitrary. A set $F \subseteq A$ is called *closed relative to* A if the complement $A \setminus F$ is relative open (if $A = \mathbb{R}$ then we call F closed).

A closed interval $[a, b]$ is closed (the complement $]-\infty, a[\cup]b, \infty[$ is open), $\mathbb{R}$ is closed (the complement $\emptyset$ is open), and $\emptyset$ is closed (the complement $\mathbb{R}$ is open). Theorem 1.28 implies

Theorem 1.30. *Let $A \subseteq \mathbb{R}$ be arbitrary. We have the following properties of relative closed sets.*

1. *If $(F_j)_{j\in J}$ is an arbitrary collection of relative closed subsets of A then their intersection $\bigcap_{j\in J} F_j$ is relative closed.*

2. *If $F_1, F_2, \ldots, F_n$ are relative closed subsets of A then their union $F_1 \cup F_2 \cup \cdots \cup F_n$ is relative closed.*

We can test if a set set is closed using convergent sequences:

Theorem 1.31. *The following is equivalent for a set $F \subseteq A \subseteq \mathbb{R}$:*

1. *F is closed relative to A.*

2. *If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence in F and $\lim_{n\to\infty} x_n \in A$ then $\lim_{n\to\infty} x_n \in F$.*

Proof. Assume F is closed relative to A and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in F. Assume $x = \lim_{n\to\infty} x_n \in A \setminus F$. Because $A \setminus F$ is open relative to A we can find $r > 0$ such that $B(x, r) \cap A \subseteq A \setminus F$, i.e., $F \cap B(x, r) = \emptyset$. We can choose $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow |x_n - x| < r$, but then $x_n \in B(x, r) \cap F$, a contradiction.

Conversely, assume that 2. holds we need to show that $A \setminus F$ is open relative to A. Assume the opposite. Then we can find $x \in A \setminus F$ such that we for all $r > 0$ have that $B(x, r) \cap A \not\subseteq A \setminus F$, i.e., that $B(x, r) \cap F \neq \emptyset$. For $n \in \mathbb{N}$ we now choose $x_n \in B(x, \frac{1}{n}) \cap F$. We have $|x_n - x| < \frac{1}{n}$ so $x_n \to x \in A$ for $n \to \infty$. But then $x \in F$, a contradiction. $\square$

1.5 Exercises

Exercise 1.1. Is the sequence $\left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$ convergent? If that is the case what is the limit?

Exercise 1.2. Is the sequence $\left(\frac{n^2+n}{n^2}\right)_{n\in\mathbb{N}}$ convergent? If that is the case what is the limit?

Exercise 1.3. Is the sequence $\left(\frac{n^3+n}{n^2}\right)_{n\in\mathbb{N}}$ convergent? If that is the case what is the limit?

Exercise 1.4. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and assume that $x_n \neq 0$ for all $n \in \mathbb{N}$. Show:

- If $x_n \to 0$ for $n \to \infty$ then $\frac{1}{x_n} \to \infty$ for $n \to \infty$.
- If $x_n \to \infty$ for $n \to \infty$ then $\frac{1}{x_n} \to 0$ for $n \to \infty$.

Exercise 1.5. Consider the sequence $((-1)^n)_{n\in\mathbb{N}}$. Can you find a convergent subsequence?

Exercise 1.6. Consider the sequence $(\frac{p_n}{q_n})_{n\in\mathbb{N}}$ in $\mathbb{Q}$ from Example 1.7 and let $x \in \mathbb{R}$. Show that there is a subsequence $(\frac{p_{n_k}}{q_{n_k}})_{k\in\mathbb{N}}$ such that $\frac{p_{n_k}}{q_{n_k}} \to x$ for $k \to \infty$. Hint: Use that any rational number appears infinitely many times in the sequence and that x can be approximated arbitrarily well by a rational number.

Exercise 1.7. Let $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\}$. Show that A is bounded from both above and below. Find $\inf A$ and $\sup A$. Does A have a maximum and/or a minimum?

Exercise 1.8. Show that an open interval is open.

Exercise 1.9. Show that the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is relative closed to $]0, 1]$. But not closed in $\mathbb{R}$.

Chapter 2

Functions of one real variable

Before we move onto (vector) function of several variables we will make the concepts of continuity and differentiability precise for functions of one variable. We will also prove some important theorems about continuous functions. In particular that image of a closed interval is a closed interval.

2.1 Continuity

Definition 2.1. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called *continuous at a point* $x_0 \in I$ if we for all positive numbers ϵ can find a positive number δ such that if $x \in I$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. With logical symbols this can be written

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in I : |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon \,. \tag{2.1}$$

See Figure 2.1

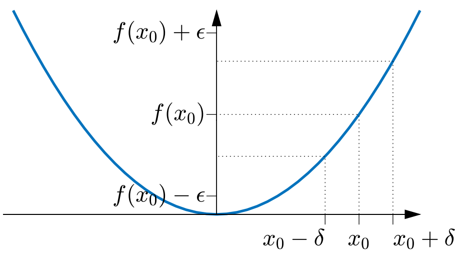

Figure 2.1: Continuity: Given an ϵ-interval around $f(x_0)$ there exits a δ-interval around x_0 that maps into the given interval around $f(x_0)$.

Remark 2.2. The continuity condition is often written as

$$f(x) \to f(x_0) \text{ for } x \to x_0 \,. \tag{2.2}$$

This notation is due to the fact that we can formulate continuity in terms of convergent sequences

Theorem 2.3. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is continuous in $x_0 \in I$ if an only if we for all sequences $(x_n)_{n\in\mathbb{N}}$ in I have that $x_n \to x_0$ for $n \to \infty$ implies that $f(x_n) \to f(x_0)$ for $n \to \infty$.*

Proof. Suppose f is continuous in x_0 and that $x_n \to x_0$ for $n \to \infty$. Given $\epsilon > 0$. As f is continuous in x_0 we can find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. As $x_n \to x_0$ for $n \to \infty$ we can find n_0 such that $n > n_0 \Rightarrow |x_n - x_0| < \delta$, but then we have $|f(x_n) - f(x_0)| < \epsilon$, i.e., $f(x_n) \to f(x_0)$ for $n \to \infty$.

Now suppose f is not continuous in x_0. Then we can find an $\epsilon > 0$ such that we for all $\delta > 0$ have some $x \in I$ with $|x - x_0| < \delta$ and $f(x) - f(x_0)| > \epsilon$. Hence, for any $n \in \mathbb{N}$ we can find $x_n \in I$ such that $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| > \epsilon$. We now have $x_n \to x_0$ for $n \to \infty$, but $f(x_n) \not\to f(x_0)$ for $n \to \infty$. □

Example 2.1. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ is continuous at all points $x \in \mathbb{R}$: For any $x_0 \in \mathbb{R}$ we have $|f(x) - f(x_0)| = |x - x_0|$. So if we have an $\epsilon > 0$ and $0 < \delta \leq \epsilon$ then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

Example 2.2. Let $c \in \mathbb{R}$ then the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = c$ is continuous at all points $x \in \mathbb{R}$: For any $x_0 \in \mathbb{R}$ we have $|f(x) - f(x_0)| = 0$. So if we have an $\epsilon > 0$ and $0 < \delta$ then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$.

Example 2.3. Let $a, b \in \mathbb{R}$ then the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax + b$ is continuous at all points $x \in \mathbb{R}$. Indeed, for any $x_0 \in \mathbb{R}$ we have $|f(x) - f(x_0)| = |ax + b - (ax_0 + b)| = |a||x - x_0|$. So if we have an $\epsilon > 0$ and $0 < \delta \leq \frac{\epsilon}{1+|a|}$ then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{|a|}{1+|a|}\epsilon < \epsilon$.

Example 2.4. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous at all points $x \in \mathbb{R}$. Indeed, for any $x_0 \in \mathbb{R}$ we have

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x + x_0||x - x_0| \leq (|x| + |x_0|)|x - x_0|\,.$$

So if we have an $\epsilon > 0$ and $0 < \delta \leq \min\{\frac{\epsilon}{1+2|x_0|}, 1\}$ then $|x| \leq |x_0| + 1$ and hence $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{|x|+|x_0|}{1+2|x_0|}\epsilon \leq \epsilon$. Observe that δ depends on x_0. The larger the x_0 is, the smaller δ needs to be.

We will use the next example later so we formulate it as a lemma.

Lemma 2.4. *The function* $\mathrm{inv} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ *given by* $\mathrm{inv}(x) = \frac{1}{x}$ *is continuous at all points* $x \in \mathbb{R} \setminus \{0\}$.

Proof. If $x_0 \in \mathbb{R}$ and $\epsilon > 0$ then we put $\delta = \min\left\{\frac{|x_0|}{2}, \frac{|x_0|^2}{2}\epsilon\right\}$. If $|x - x_0| < \delta$ then $|x| > \frac{1}{2}|x_0|$ and we have

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{xx_0}\right| = \frac{|x - x_0|}{|x||x_0|} < \frac{|x - x_0|}{\frac{1}{2}|x_0|^2} < \frac{\frac{1}{2}|x_0|^2\epsilon}{\frac{1}{2}|x_0|^2} = \epsilon\,. \qquad \square$$

Notice that the positive number δ is not unique. On the contrary, if one δ works then any smaller positive number also works.

We do not want to go through arguments as above for any conceivable function and the following theorem helps us to avoid that.

Theorem 2.5. *If $x_0 \in I$ and $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous in x_0 then*

1. *The function $f + g : I \to \mathbb{R} : x \mapsto f(x) + g(x)$ is continuous in x_0.*

2. *The function $fg : I \to \mathbb{R} : x \mapsto f(x)g(x)$ is continuous in x_0.*

If $x_0 \in I$, $f : I \to \mathbb{R}$, $f(x) \neq 0$ for all $x \in I$, and f is continuous in x_0 then

3. *The function $\frac{1}{f} : I \to \mathbb{R} : x \mapsto \frac{1}{f(x)}$ is continuous in x_0.*

If $I, J \subseteq \mathbb{R}$ are intervals, $f : I \to \mathbb{R}$ is continuous in $x_0 \in I$, $f(I) \subseteq J$, and $g : J \to \mathbb{R}$ is continuous in $y_0 = f(x_0)$ then

4. *The function $g \circ f : I \to \mathbb{R} : x \mapsto g(f(x))$ is continuous is x_0.*

Statement 1, 2, and 3 are a special cases of Theorem 3.9 and Statement 4 is a special case of Theorem 3.8 so we do not need to give the proofs here. But using the formulation in terms of sequences it is not hard:

Proof. If f and g are continuous in x_0 and $x_n \to x_0$ for $n \to \infty$ then by Theorem 2.3 $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$ for $n \to \infty$ and by Theorem 1.9 $f(x_n) + g(x_n) \to f(x_0) + g(x_0)$ but then Theorem 2.3 shows that $f + g$ is continuous in x_0. This proves Case 1. The other cases are similar. □

Using induction it is not hard to show

Theorem 2.6. *If $x_0 \in I$ and the functions $f_k : I \to \mathbb{R}$, $k = 1, 2, \dots, n$ are continuous in x_0 then*

1. *The function $f_1 + f_2 + \cdots + f_n : I \to \mathbb{R} : x \mapsto f_1(x) + f_2(x) + \cdots + f_(x)$ is continuous in x_0.*

2. *The function $f_1 f_2 \cdots f_n : I \to \mathbb{R} : x \mapsto f_1(x)f_2(x) \cdots f_n(x)$ is continuous in x_0.*

The examples we have seen so far have all been continuous at all points in the domain. In that case we call the function continuous:

Definition 2.7. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called *continuous* if it is continuous at all points $x \in \mathbb{R}$. With logical symbols this can be written

$$\forall x \in I \, \forall \epsilon > 0 \, \exists \delta > 0 \, \forall y \in I : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \,. \tag{2.3}$$

Example 2.5. Any polynomial $p : \mathbb{R} \to \mathbb{R}$, $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous. Indeed, we know the functions $f : x \mapsto x$ and $g : x \mapsto a_k$ are continuous so the second statement in Theorem 2.6 tells us that the functions $x \mapsto a_k x^k$ are continuous for all $k = 0, 1, \ldots, n$. The first statement in the theorem now tells us that p is continuous.

Example 2.6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x\,, & x \in \mathbb{Q}\,, \\ 0\,, & x \in \mathbb{R} \setminus \mathbb{Q}\,. \end{cases}$$

Then f is continuous in 0 only.

We will without proof use that the trigonometric functions cos, sin, tan, cot, their inverses arccos, arcsin, arctan, arccot, the exponential function exp, and the natural logarithm log are continuous functions.

If $a > 0$ then $a^x = \exp(\log(a)x)$ so Theorem 2.6 tells us that $x \mapsto a^x$ is continuous. Similar $\log_a(x) = \frac{\log(x)}{\log(a)}$ so $\log_a$ is continuous. If $x > 0$ then $x^a = \exp(a \log(x))$ so $x \mapsto x^a$ is continuous.

We can formulate continuity in terms of open or closed sets

Theorem 2.8. *let $A \subseteq \mathbb{R}$ be arbitrary and let $f : A \to \mathbb{R}$. The following is equivalent*

1. *The function f is continuous.*
2. *For all open sets $U \subseteq \mathbb{R}$ the preimage $f^{-1}(U)$ is relative open in A.*
3. *For all closed sets $F \subseteq \mathbb{R}$ the preimage $f^{-1}(F)$ is relative closed in A.*

Proof. $1 \Rightarrow 2$: Suppose f is continuous, $U \subseteq \mathbb{R}$ is open, and $x_0 \in f^{-1}(U)$. As U is open we can choose $r > 0$ such that $]f(x_0) - r, f(x_0) + r[\subseteq U$. As f is continuous we can choose $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < r$. We now have $f(A\cap]x_0 - \delta, x_0 + \delta[) \subseteq]f(x_0) - r, f(x_0) + r[\subseteq U$. Hence $A\cap]x_0 - \delta, x_0 + \delta[\subseteq f^{-1}(U)$. By Theorem 1.27 $f^{-1}(U)$ is relative open.

$2 \Rightarrow 1$: Suppose $x_0 \in A$ and $\epsilon > 0$. The interval $]f(x_0) - \epsilon, f(x_0) + \epsilon[$ is open so by assumption $f^{-1}(]f(x_0) - \epsilon, f(x_0) + \epsilon[)$ is relative open in A. By Theorem 1.27 we can find $\delta > 0$ such that $A\cap]x_0 - \delta, x_0 + \delta[\subseteq f^{-1}(]f(x_0) - \epsilon, f(x_0) + \epsilon[)$. But then $f(A\cap]x_0 - \delta, x_0 + \delta[) \subseteq]f(x_0) - \epsilon, f(x_0) + \epsilon[$, i.e., we have found a $\delta > 0$ such that we for $x \in A$ have that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. As ϵ was arbitrary f is continuous in x_0 and as x_0 was arbitrary f is continuous.

$2 \Rightarrow 3$: Suppose f satisfies 2 and that $F \subseteq \mathbb{R}$ is closed. Then $\mathbb{R} \setminus F$ is open $A \setminus f^{-1}(F) = f^{-1}(\mathbb{R} \setminus F)$ is relative open, i.e., $f^{-1}(F)$ is relative closed.

$3 \Rightarrow 2$: If f satisfies 3 and $U \subseteq \mathbb{R}$ is open then $\mathbb{R} \setminus U$ is closed and hence $f^{-1}(\mathbb{R} \setminus U)$ is relative closed. Now $f^{-1}(U) = A \setminus f^{-1}(\mathbb{R} \setminus U)$ is relative open. □

The following theorem is very important for mathematical analysis.

Theorem 2.9. *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous. Then the following holds:*

1. *The image $f(I)$ is an interval. We also call sets on the following form for intervals: $]-\infty,\infty[$, $]-\infty,b]$, and $]a,\infty[$, where $a,b\in\mathbb{R}$.*

2. *If I is closed, i.e., $I=[a,b]$, where $a,b\in\mathbb{R}$, then $f(I)$ is closed, i.e., $f(I)=[c,d]$, where $c,d\in\mathbb{R}$.*

Proof. Part 1: Let I be an interval and let $f:I\to\mathbb{R}$ be a continuous function. We want to show that $f(I)$ is an interval (allowing $\pm\infty$ as endpoints). We do it indirectly, i.e., we assume the opposite and arrive at a contradiction.

So assume $f(I)$ is not an interval. Then there exists $c\in\mathbb{R}$ and $a,b\in I$ such that $f(a)<c<f(b)$ and $c\notin f(I)$.

First assume that $a<b$. As $f(a)<c$, $f(b)>c$, and f is continuous there exists $\delta_1,\delta_2>0$ such that

$$f(]a,a+\delta_1[)\subseteq]-\infty,c[)\quad\text{and}\quad f(]b-\delta_2,b[)\subseteq]c,\infty[)\,. \tag{2.4}$$

Put $U=]a,b[\cap f^{-1}(]-\infty,c[)$ and $V=]a,b[\cap f^{-1}(]c,\infty[)$. They are open sets and (2.4) shows that $]a,a+\delta_1[\subseteq U$ and $]b-\delta_2,b[\subseteq V$. Furthermore, $]a,b[=U\cup V$, and $U\cap V=\emptyset$.

Put $d=\sup U$, then $a<d<b$. The set U is open so if $d\in U$ then we can find $\epsilon>0$ such that $]d-\epsilon,d+\epsilon[\subseteq U$, but that contradicts that $d=\sup U$. Hence $d\notin U$. The set V is open so if $d\in V$ then we can find $\epsilon>0$ such that $]d-\epsilon,d+\epsilon[\subseteq V$, but as $]d-\epsilon,d]\cap U\neq\emptyset$ that contradicts that $U\cap V=\emptyset$. Hence $d\notin V$.

We see that we must have $d\notin U\cup V=]a,b[$, but that contradicts that $a<d<b$. That means the existence of c is impossible. Hence $f(I)$ has to be an interval and we have proved the first statement in Theorem 2.9 in the case $a<b$.

If $b>a$ we can look at $-f$ and then we see that $-f(b)<-f(a)$ and the first case shows that $-f(I)$ is an interval, but then $f(I)$ is an interval too.

Part 2: Let $I=[a,b]$ and let $f:I\to\mathbb{R}$ be a continuous function. We know $f(I)$ is an interval and we want to show that it is a closed interval.

We first show that $f(I)$ is bounded. Assume the opposite. Then we can find $x_n\in[a,b]$ such that $|f(x_n)|\to\infty$ for $n\to\infty$. By Theorem 1.13 we have a convergent subsequence $x_{n_k}\to x$ for $k\to\infty$. As $x_{n_k}\in[a,b]$ Lemma 1.7 shows that $x\in[a,b]$, but now we have $f(x_{n_k})\to f(x)$ and also $|f(x_{n_k})|\to\infty$, a contradiction.

We now know we have $c,d\in\mathbb{R}$ such that $]c,d[\subseteq f(I)\subseteq[c,d]$. We can find a sequence $(x_n)_{n\in\mathbb{N}}$ is $[a,b]$ such that $f(x_n)\to d$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is bounded so by Corollary 1.13 it has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Let $x=\lim_{k\to\infty}x_{n_k}$. We have $x\in[a,b]$ and as f is continuous $f(x)=\lim_{k\to\infty}f(x_{n_k})=d$, i.e., $d\in f(I)$. In exactly the same manner we can show that $c=\in f(I)$. Hence $f(I)=[c,d]$. □

As an easy consequence we have

Corollary 2.10. *Let $a,b\in\mathbb{R}$ and let $f:[a,b]\to\mathbb{R}$ be continuous. Then f attains its maximum and minimum, i.e., there exist $x_1,x_2\in[a,b]$ such that $f(x_1)\le f(x)\le f(x_2)$ for all $x\in[a,b]$.*

Proof. By Theorem 2.9 we have that $f(I) = [c, d]$ for some $c, d \in \mathbb{R}$, but then $c = \min_{x \in I} f(x)$ and $d = \max_{x \in I} f(x)$. So there exist $x_1, x_2 \in [a, b]$ such that $f(x_1) = c$ and $f(x_2) = d$. □

In Example 2.4 we saw that given an $\epsilon > 0$ we may not been able to find a $\delta > 0$ that works for all $x \in I$. If that is possible we call the function uniformly continuous. The precise definition is

Definition 2.11. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called *uniformly continuous* if we for all positive numbers ϵ can find a positive number δ such that if $x, y \in I$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. With logical symbols this can be written

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x, y \in I : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon\,. \tag{2.5}$$

Example 2.7. In example 2.3, where $f(x) = ax + b$, we saw that if $\epsilon > 0$ and $0 < \delta < \frac{\epsilon}{1+|a|}$ then $|x - y| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. So f is uniformly continuous.

Example 2.8. The function $f; \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is *not* uniformly continuous. Indeed, let $\epsilon = 1$ and let $\delta > 0$. If $x > 1/\delta$ and $y = x + \delta/2$ then $|x - y| < \delta$, but $|f(x) - f(y)| = |2x\delta + \delta^2| > 1 + \delta^2 > \epsilon$.

The following theorem says that a continuous function on a bounded closed interval is uniformly continuous. It is essential for the definition of the Riemann integral, see Section 2.3.

Theorem 2.12. *If $f : [a, b] \to \mathbb{R}$ is continuous then f is uniformly continuous.*

Proof. Suppose the opposite, i.e., that f is not uniformly continuous. Then there exists an $\epsilon > 0$ and for each $n \in \mathbb{N}$ can we find $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \epsilon$. By Theorem 1.13 we can find a a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$, let $x_0 = \lim_{k \to \infty} x_{n_k}$ as $|x_{n_k} - x_{n_k}| < \frac{1}{n_k}$ we also have $y_{n_k} \to x_0$ for $k \to \infty$. As f is continuous we can find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon/2$. We can now find $k_1 \in \mathbb{N}$ such that $k > k_1 \Rightarrow |x_0 - x_{n_k}| < \delta$ and $k_2 \in \mathbb{N}$ such that $k > k_2 \Rightarrow |x_0 - y_{n_k}| < \delta$. If we put $k_0 = \max\{k_1, k_2\}$ then

$$\begin{aligned} k > k_0 \implies |x_0 - x_{n_k}|, |x_0 - y_{n_k}| < \delta \implies \\ |f(x_0) - f(x_{n_k})|, |f(x_0) - f(y_{n_k})| < \epsilon/2 \implies \\ |f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(y_{n_k})| < \epsilon/2 + \epsilon/2 = \epsilon\,, \end{aligned}$$

a contradiction. □

2.2 Differentiability

Definition 2.13. Let $I \subseteq \mathbb{R}$ be an *open* interval. A function $f : I \to \mathbb{R}$ is called *differentiable at a point* $x \in I$ if there exists a number $c \in \mathbb{R}$ such that

$$\frac{f(x + h) - f(x)}{h} \to c \quad \text{for } h \to 0\,. \tag{2.6}$$

With logical symbols this can be written

$$\exists c\, \forall \epsilon > 0\, \exists \delta > 0\, \forall h \neq 0 : |h| < \delta \implies \left| \frac{f(x+h) - f(x)}{h} - c \right| < \epsilon \,. \tag{2.7}$$

The number c is called the *derivative* at x_0.

Remark 2.14. The condition (2.6) can be rewritten as

$$\frac{f(x+h) - f(x) - ch}{h} \to 0 \quad \text{for } h \to 0 \,. \tag{2.8}$$

Remark 2.15. If we put

$$\epsilon(h) = \frac{f(x+h) - f(x) - ch}{h} \,, \tag{2.9}$$

then $\epsilon(h) \to 0$ for $h \to 0$ and we have

$$f(x+h) = f(x) + ch + \epsilon(h)h \,. \tag{2.10}$$

That is, we can approximate f around x_0 by a first degree polynomial and the error, $\epsilon(h)|h|$, goes to zero faster than h.

Remark 2.16. The condition (2.6) has a geometrical interpretation: The line through $(x, f(x_0))$ and $(x_0+h, f(x_0+h))$ (a secant) has a well defined limit position as $h \to 0$ (the *tangent* to the graph). The slope of the secant is $\frac{f(x+h)-f(x)}{h}$ and the slope of the tangent is the limit c, see Figure 2.2.

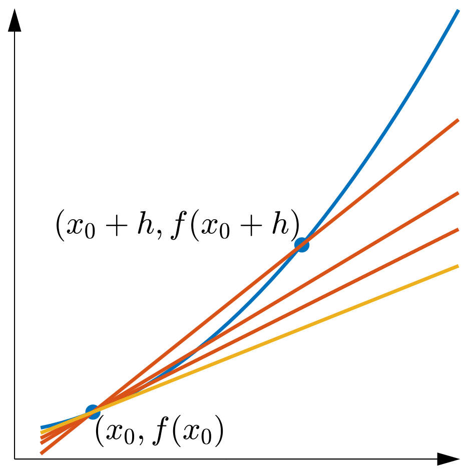

Figure 2.2: The limit position of the secant (red) as $h \to 0$ is the tangent (yellow).

Before we look at some examples we note that differentiable functions are continuous:

Theorem 2.17. *If I is an open interval and $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ then f is continuous at x_0.*

Proof. Let c be the derivative of f at x_0 and consider $x \in \mathbb{R}$ if we let $h = x - x_0$ then we have $|x - x_0| = |h|$ and

$$f(x) - f(x_0) = f(x+h) - f(x) = h\frac{f(x+h) - f(x)}{h} \underset{h\to 0}{\longrightarrow} 0 \cdot c = 0\,. \qquad \square$$

Example 2.9. The function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x$ is differentiable at all $x \in \mathbb{R}$ with derivative 1. Indeed

$$\frac{f(x+h) - f(x)}{h} = \frac{x+h-x}{h} = \frac{h}{h} = 1\,.$$

Example 2.10. Let $c \in \mathbb{R}$. The function $f : \mathbb{R} \to \mathbb{R} : x \mapsto c$ is differentiable at all $x \in \mathbb{R}$ with derivative 0. Indeed

$$\frac{f(x+h) - f(x)}{h} = \frac{c-c}{h} = \frac{0}{h} = 0\,.$$

Example 2.11. The function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ is differentiable at all points with derivative $2x$. Indeed

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h \underset{h\to 0}{\longrightarrow} 2x\,.$$

As in the previous chapter the next example is formulated as a lemma.

Lemma 2.18. *The function* $\mathrm{inv} : \mathbb{R} \setminus \{0\} \to \mathbb{R} : x \mapsto \frac{1}{x}$ *is differentiable at all points with derivative* $\frac{-1}{x^2}$.

Proof. Let $x \neq 0$ and assume $h \neq 0$ and $x + h \neq 0$. Then

$$\left|\frac{\frac{1}{x+h} - \frac{1}{x} - \frac{-1}{x^2}h}{h}\right| = \left|\frac{x^2 - x(x+h) + h(x+h)}{h(x+h)x^2}\right| = \left|\frac{h}{(x+h)x^2}\right| \to 0\,,$$

for $h \to 0$. $\square$

Theorem 2.19. *If* $x_0 \in I$ *and* $f : I \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *are differentiable at* x_0 *with derivatives* a *and* b*, respectively. Then*

1. *The function* $f + g : I \to \mathbb{R} : x \mapsto f(x) + g(x)$ *is differentiable in* x_0 *with derivative* $a + b$.

2. *The function* $fg : I \to \mathbb{R} : x \mapsto f(x)g(x)$ *is differentiable at* x_0 *with derivative* $f(x_0)a + bg(x_0)$.

If $x_0 \in I$, $f : I \to \mathbb{R}$, $f(x) \neq 0$ *for all* $x \in I$, *and* f *is differentiable at* x_0 *with derivative* c *then*

3. *The function* $\frac{1}{f} : I \to \mathbb{R} : x \mapsto \frac{1}{f(x)}$ *is differentiable at* x_0 *with derivative* $\frac{-c}{f(x_0)^2}$.

If $I, J \subseteq \mathbb{R}$ are open intervals, $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$ with derivative a, $f(I) \subseteq J$, and $g : J \to \mathbb{R}$ is differentiable in $y_0 = f(x_0)$ with derivative b then

4. *The function $g \circ f : I \to \mathbb{R} : x \mapsto g(f(x))$ is differentiable at x_0 with derivative ab.*

Statement 1, 2, and 3 are a special cases of Theorem 3.29 and statement 4 is a special case of Theorem 3.28 so we will not give the proofs here. But it is a good exercise to prove it now.

Using induction it is not hard to show

Theorem 2.20. *If $x_0 \in I$ and the functions $f_k : I \to \mathbb{R}$, $k = 1, 2, \ldots, n$ are differentiable x_0 with derivative c_k then*

1. *The function $f_1 + f_2 + \cdots + f_n : I \to \mathbb{R} : x \mapsto f_1(x) + f_2(x) + \cdots + f_(x)$ is differentiable at x_0 with derivative $c_1 + \cdots + c_n$.*

2. *The function $f_1 f_2 \cdots f_n : I \to \mathbb{R} : x \mapsto f_1(x) f_2(x) \cdots f_n(x)$ is differentiable at x_0 with derivative*

$$c_1 f_2(x_0) f_3(x_0) \ldots f_n(x_0) + f_1(x_0) c_2 f_3(x_0) \ldots f_n(x_0) + \\ \cdots + f_1(x_0) \ldots f_{n-1}(x_0) c_n \,.$$

Definition 2.21. Let $I \subseteq \mathbb{R}$ be an *open* interval. A function $f : I \to \mathbb{R}$ is called *differentiable* if it is differentiable at all points $x \in I$. The derivative at x is denoted $f'(x)$, $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$, or $f^{(1)}(x)$, and the function $f' : \mathbb{R} \to \mathbb{R} : x \mapsto f'(x)$ is called the *derivative* of f. If f' is continuous then f is called a C^1 function.

We can define higher order derivatives recursively:

Definition 2.22. Let $I \subseteq \mathbb{R}$ be an *open* interval, let $k \in \mathbb{Z}$, and $k \geq 2$. The function is called *k times differentiable* if it is differentiable and f' is $k-1$ times differentiable. The *kth derivative* is $f^{(k)} = (f')^{(k-1)}$. If $f^{(k)}$ is continuous then f is called a C^k function. If f is k times differentiable for all k then f is called a C^∞ function.

Remark 2.23. For lower order derivatives we also use the notation $f'' = f^{(2)}$, $f''' = f^{(3)}$, etc.

Example 2.12. The function $f(x) = x^n$ is differentiable with derivative nx^{n-1}. This follows from the second statement in Theorem 2.20. It can also be shown by induction on n: The case $n = 1$ is Example 2.9 and if $f(x) = x^n$ is differentiable with derivative nx^{n-1} and $g(x) = x$ then $x^{n+1} = f(x)g(x) = (fg)(x)$ and the second statement in Theorem 2.19 now tell us that $fg : x \mapsto x^{n+1}$ is differentiable with derivative

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) = nx^{n-1}x + x^n \cdot 1 = (n+1)x^n \,.$$

A similar argument shows that $x \mapsto x^n$ is a C^∞ function.

Example 2.13. Any polynomial $p : \mathbb{R} \to \mathbb{R}$, $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is differentiable with derivative $p'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$. This follows from the previous example and the first statement in Theorem 2.20. Again induction on the degree of the polynomial shows that a polynomial is a C^∞ function.

Definition 2.24. We say that a function $f : I \to \mathbb{R}$ has a *local maximum* at $x_0 \in I$ if there exist a $r > 0$ such that $f(x) \leq f(x_0$ for all $x \in I \cap [x_0 - r, x_0 + r]$.

We say that f has a *local minimum* at $x_0 \in I$ if there exist a $r > 0$ such that $f(x) \leq f(x_0$ for all $x \in I \cap [x_0 - r, x_0 + r]$.

Lemma 2.25. *Suppose $f : I \to \mathbb{R}$ is differentiable in $x_0 \in I$ and has a local maximum or minimum in x_0 then $f'(x_0) = 0$.*

Proof. For $h \neq 0$ we put

$$\epsilon(h) = \frac{f(x_0 + h) - f(x_0) - f'(x_0)}{h} .$$

If $f'(x_0) > 0$ then we can find $\delta > 0$ such that $|h| < \delta \Rightarrow |\epsilon(h)| < f'(x_0)$. For $|h| < \delta$ we now have $f'(x_0) + \epsilon(h) > 0$ and that implies that

$$f(x_0 + h) = \begin{cases} f(x_0) + (f'(x_0) + \epsilon(h))h > f(x_0) , & 0 < h < \delta , \\ f(x_0) + (f'(x_0) + \epsilon(h))h < f(x_0) , & -\delta < h < 0 . \end{cases}$$

So $f(x_0)$ is neither a local minimum nor a local maximum. If $f'(x_0) < 0$ then $-f'(x_0) > 0$ so $-f(x_0)$ is neither a local minimum nor a local maximum for $-f$, but then the same is true for f. Hence we must have $f'(x_0) = 0$. □

Example 2.14. The opposite is not true: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ has the derivative $f'(x) = 3x^2$ so $f'(0) = 0$. But if $x < 0$ then $f(x) < 0 = f(0)$ and if $x > 0$ then $f(x) > 0 = f(0)$ so $f(0)$ is neither a local minimum nor a local maximum.

Lemma 2.26 (Rolle's theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous, f is differentiable on the open interval $]a, b[$, and $f(a) = f(b)$ then there exist $\xi \in]a, b[$ such that $f'(\xi) = 0$.*

Proof. By Corollary 2.10 f attains it minimum and maximum. If the minimum or maximum is attained in the open interval $]a, b[$ then Lemma 2.25 yields the result. Otherwise we have the minimum and the maximum at one of the endpoints a and b, but as $f(a) = f(b)$ the minimum is the same as the maximum. So f must be constant and consequently $f'(x) = 0$ for all $x \in]a, b[$. □

We can now show the mean value theorem:

Theorem 2.27 (Mean value theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on the open interval $]a, b[$ then there exist $\xi \in]a, b[$ such that*

$$f(b) - f(a) = f'(\xi)(b - a) . \tag{2.11}$$

Proof. Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - \frac{f(a)(b-x) + (x-a)f(b)}{b-a} \,.$$

Then g is continuous, differentiable on $]a, b[$ and with derivative $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. Furthermore, $g(a) = g(b) = 0$. So by Rolle's theorem we have a $\xi \in]a, b[$ such that $g'(\xi) = 0$. But then $f'(\xi) = \frac{f(b)-f(a)}{b-a}$ which is equivalent to (2.11). □

Remark 2.28. We have a geometrical interpretation of the theorem: The graph of f has a tangent that is parallel to the line between the end points, see Figure 2.3 left.

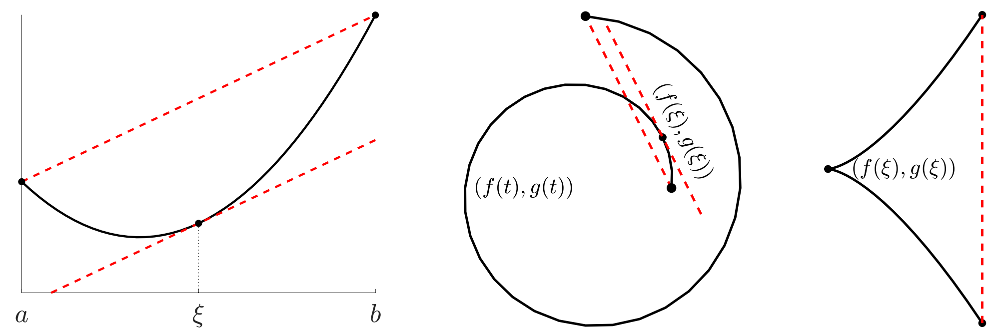

Figure 2.3: Left: A differentiable function have a tangent parallel to the line between the end points. Middle: A smooth curve has a tangent parallel to the line between the end points. Right: If a curve has a cusp (where $f'(\xi), g'(\xi) = (0, 0)$) there need not be a tangent parallel to the line between the end points.

We easily obtain the following variant

Theorem 2.29. *Let $I \subseteq \mathbb{R}$ be an open interval, let $f : I \to \mathbb{R}$ be differentiable and let $x, x+h \in I$ then there exist ξ between x and $x+h$ such that*

$$f(x+h) = f(x) + f'(\xi)h \,. \tag{2.12}$$

Proof. If $h > 0$ we but $a = x$ and $b = x + h$ and if $h < 0$ we put $a = x - h$ and $b = x$. In both cases the result is the same as Theorem 2.27. □

As a corollary we have

Corollary 2.30. *Let $I \subseteq \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be differentiable. If $f'(x) = 0$ at all points $x \in I$ then f is constant.*

Proof. Let $x_0, x \in I$ and put $h = x - x_0$. Then the Mean value theorem says that $f(x) = f(x_0 + h) = f(x_0) + f'(\xi)h = f(x_0)$. □

In order to prove Taylor's theorem with reminder we need a slightly stronger version of the mean value theorem.

Theorem 2.31 (Cauchy's mean value theorem)**.** *If $f, g : [a, b] \to \mathbb{R}$ are continuous and differentiable on the open interval $]a, b[$ then there exist $\xi \in]a, b[$ such that*

$$(f(b) - f(a))g'(\xi) = f'(\xi)(g(b) - g(a)) . \tag{2.13}$$

If $g(a) \neq g(b)$ we have

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} . \tag{2.14}$$

Proof. If $g(a) = g(b)$ then Rolle's theorem yields $\xi]a, b[$ such that $g'(\xi) = 0$ and both sides of (2.13) is zero.

If $g(a) \neq g(b)$ then we define $h : [a, b] \to \mathbb{R}$ by $h(x) = f(x) - g(x)\frac{f(b)-f(a)}{g(b)-g(a)}$. We see that

$$h(b) - h(a) = f(b) - f(a) - (g(b) - g(a))\frac{f(b) - f(a)}{g(b) - g(a)} = 0 .$$

As h is continuous and differentiable on the open interval $]a, b[$ Rolle's theorem give us a $\xi \in]a, b[$ such that $h'(\xi) = 0$, i.e., $f'(\xi) = g'(\xi)\frac{f(b)-f(a)}{g(b)-g(a)}$. Multiplying with $g(b) - g(a)$ yields (2.13) and multiplying with $\frac{1}{g'(\xi)}$ yields (2.14). □

Remark 2.32. If we let $g(x) = x$ then we obtain the ordinary mean value theorem.

Remark 2.33. If $(f'(t), g'(t)) \neq (0, 0)$ for all $t \in]a, b[$ we have a geometrical interpretation of the theorem: The curve $(f(t), g(t))$ has a tangent that is parallel to the line between the end points $(f(a), g(a))$ and $(f(b), g(b))$, see Figure 2.3 middle and right.

We can generalise (2.10) to higher order derivatives, but first we prove the following Lemma.

Lemma 2.34. *Let $I \subseteq \mathbb{R}$ be an* open *interval, let $f : I \to \mathbb{R}$ be an n times differentiable function and let $x, x + h \in I$. If $f^{(k)}(x) = 0$ for $k = 0, 1, \ldots, n$ then*

$$\frac{f(x + h)}{h^n} \to 0 \quad \text{for } h \to 0 . \tag{2.15}$$

Proof. The proof is by induction on n. The case $n = 1$ is the definition of differentiability (when $f(x), f'(x) = 0$). So assume the theorem holds for an $n \in \mathbb{N}$, that f is $n + 1$ times differentiable, and that $f^{(k)}(x) = 0$ for $k = 0, 1, \ldots, n + 1$. Then f' satisfies the conditions in the theorem so

$$\frac{f'(x + h)}{h^n} \to 0 \quad \text{for } h \to 0 .$$

By the mean value theorem we have $f(x + h) = f'(x + \xi h)$ for a $\xi \in]0, 1[$ and hence

$$\left|\frac{f(x + h)}{h^{n+1}}\right| = \left|\frac{f'(x + \xi h)}{h^n}\right| \leq \left|\frac{f'(x + \xi h)}{(\xi h)^n}\right| \to 0 \quad \text{for } h \to 0 .$$ □

Theorem 2.35 (Taylor's theorem)**.** *Let $I \subseteq \mathbb{R}$ be an* open *interval, let $f : I \to \mathbb{R}$ be an n times differentiable function and let $x, x+h \in I$. Then*

$$f(x+h) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) h^k + \epsilon(h)|h|^n$$
$$= f(x) + f'(x)h + \frac{f''(x)}{2} h^2 + \cdots + \frac{f^{(n)}(x)}{n!} h^n + \epsilon(h) h^n \,, \quad (2.16)$$

where $\epsilon(h) \to 0$ for $h \to 0$.

Proof. If we put $g(t) = f(x+t) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) t^k$ then $g^{(k)}(0) = 0$ for $k = 0, \ldots, n$. So Lemma 2.34 tells us that

$$\epsilon(h) = \frac{g(x+h)}{h^n} \to 0 \quad \text{for } h \to 0 \,. \qquad \square$$

Remark 2.36. The polynomial $\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) h^k$ is called the nth degree *Taylor polynomial*.

We can also generalise the mean value theorem to higher order derivatives, but here we need the following lemma.

Lemma 2.37. *Let $I \subseteq \mathbb{R}$ be an* open *interval, let $f : I \to \mathbb{R}$ be an n times differentiable function and let $x, x+h \in I$. If $f^{(k)}(x) = 0$ for $k = 0, 1, \ldots, n$ then there exists $\xi \in]0, 1[$ such that*

$$f(x+h) = \frac{f^{(n)}(x+\xi h)}{n!} h^n \,. \quad (2.17)$$

Proof. As $f(x) = 0$ (2.17) is obvious if $h = 0$. So assume $h \neq 0$. Then (2.17) is equivalent to $\frac{f(x+h)}{h^n} = \frac{f^{(n)}(x+\xi h)}{n!}$. If we put $g(t) = (t-x)^n$ then $g(x+h) = h^n$ and $g^{(n)}(t) = n!$. As $f(x) = g(x) = 0$ we can rewrite the equation as

$$\frac{f(x+h) - f(x)}{g(x+h) - g(x)} = \frac{f(x+h)}{g(x+h)} = \frac{f^{(n)}(x+\xi h)}{g^{(n)}(x+\xi h)} \,.$$

We will prove this using induction on n. The case $n = 1$ is Cauchy's mean value theorem (Lemma 2.31). Now assume the theorem is true for an $n \in \mathbb{N}$ and that $f : I \to \mathbb{R}$ is an n times differentiable function with $f^{(k)}(x) = 0$ for $k = 0, 1, \ldots, n+1$. The function f is differentiable so Cauchy's mean value theorem yields a $\xi_1 \in]0, 1[$ such that $\frac{f(x+h)}{g(x+h)} = \frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f'(x_0+\xi_1 h)}{g'(x_0+\xi_1 h)}$. The functions f' and g' are n times differentiable so by the induction hypothesis we can find $\xi_2 \in]0, 1[$ such that $\frac{f'(x_0+\xi_1 h)}{g'(x_0+\xi_1 h)} = \frac{f^{(n+1)}(x+\xi_2\xi_1 h)}{g^{(n+1)}(x+\xi_2\xi_1 h)}$. Letting $\xi = \xi_2 \xi_1$ we have $\frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f^{(n+1)}(x+\xi h)}{g^{(n+1)}(x+\xi h)}$ and we are done. $\square$

This lemma gives us the following generalisation of Theorem 2.29:

Theorem 2.38 (Taylor's theorem with reminder). *Let $I \subseteq \mathbb{R}$ be an* open *interval, let $f : I \to \mathbb{R}$ be a C^n function and let $x, x+h \in I$. Then there exist $\xi \in]0,1[$ such that*

$$f(x+h) = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x) h^k + \frac{1}{n!} f^{(n)}(x+\xi h) h^n \,. \tag{2.18}$$

Proof. If we put $g(t) = f(x+t) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) t^k$ then $g^{(k)}(0) = 0$ for $k = 0, \dots, n$. So Lemma 2.37 gives us a $\xi \in]0,1[$ such that $g(h) = g^{(n)}(\xi h) h^n$. Furthermore, $g^{(n)}(t) = f^{(n)}(x+t) - f^{(n)}(x) = f^{(n)}(x+t)$ and

$$f(x+h) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) h^k + g(h) = \sum_{k=0}^{n-1} \frac{1}{k!} f^{(k)}(x) h^k + \frac{1}{n!} f^{(n)}(x+\xi h) h^n \,. \qquad \square$$

We saw in Lemma 2.25 that $f'(x_0) = 0$ was a necessary condition for having a local maximum or minimum, but in Example 2.14 we also saw that it is not sufficient. If the second derivative is non zero then we can say more:

Theorem 2.39. *Let $I \subseteq \mathbb{R}$ be an open interval and let $x_0 \in I$. Suppose $f : I \to \mathbb{R}$ is twice differentiable with $f'(x_0) = 0$. If $f''(x_0) > 0$ then $f(x_0)$ is a local maximum and if $f''(x_0) < 0$ then $f(x_0)$ is a local minimum*

Proof. By Taylor's theorem we have

$$\begin{aligned} f(x_0+h) &= f(x_0) + f'(x_0)h + \frac{1}{2} f''(x_0) h^2 + \epsilon(h) h^2 \\ &= f(x_0) + \left(\frac{1}{2} f''(x_0) + \epsilon(h) \right) h^2 \,, \end{aligned}$$

where $\epsilon(h) \to 0$ for $h \to 0$. We can find $r > 0$ such that $|h| < r \Rightarrow \epsilon(h) < \frac{1}{2}|f''(x_0)|$. As $h^2 \geq 0$ we see that $\left(\frac{1}{2} f''(x_0) + \epsilon(h)\right) h^2$ has the same sign as $f''(x_0)$ for $|h| < r$ and now the result follows. $\square$

Example 2.15. Suppose we want to find local maxima and minima for the polynomial $f(x) = x^3 - 3x$. The derivative is $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x+1)(x-1)$. So the potential local minima and maxima are $x = \pm 1$. The second derivative is $f''(x) = 6x$ so $f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$ so we have a local maximum for $x = -1$ and a local minimum for $x = 1$, see Figure 2.4.

Example 2.16. Suppose we want to find local maxima and minima for the polynomial $f(x) = x^4 - 2x^2$. The derivative is $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x+1)(x-1)$. So the potential local minima and maxima are $x = -1, 0, 1$. The second derivative is $f''(x) = 12x^2 - 4$ so $f''(\pm 1) = 12 - 4 = 8 > 0$ so we have a local minimum for $x = \pm 1$. We have $f''(0) = -4$ so we have a local maximum for $x = 0$, see Figure 2.4.

Example 2.17. Suppose we want to find local maxima and minima for the polynomial $f(x) = \frac{1}{5}x^5 - \frac{1}{3}x^3$. The derivative is $f'(x) = x^4 - x^2 = x^2(x^2-1) = 4x(x+1)(x-1)$. So the potential local minima and maxima are $x = -1, 0, 1$. The second derivative is $f''(x) = 4x^3 - 2x$ so $f''(-1) = -4 + 2 = -2$ and $f''(1) = 4 - 2 = 2$. So we have a local maximum for $x = -1$ and a local minimum for $x = 1$. The third potential point is $x = 0$ but $f''(0) = 0$ and we cannot say anything. (The third derivative is $f'''(0) = -2$ and it can be shown that this implies that we neither have a maximum nor a minimum), see Figure 2.4.

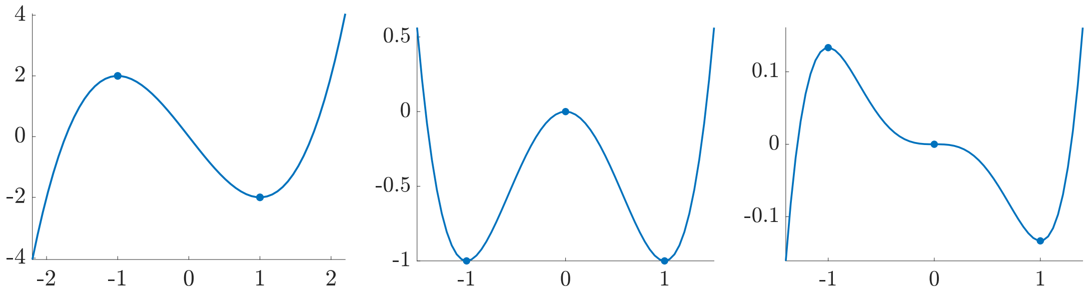

Figure 2.4: Left to right: Example 2.15, 2.16, and 2.17.

It is in general not so easy to find the zeros of f' and more often than not we have to resort to numerical methods.

The following is a special case of the *inverse function theorem*.

Theorem 2.40 (Inverse function theorem). *Let $I \subseteq \mathbb{R}$ be an open interval and let $f : I \to \mathbb{R}$ be differentiable with continuous derivative $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Then the image $J = f(I)$ is an open interval, f is invertible, and the inverse $f^{-1} : J \to I$ is differentiable with derivative $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$.*

Proof. As f is continuous $J = f(I)$ is an interval. We need to show that it is open.

As f' is continuous $f'(I)$ is an interval and as $0 \notin f'(I)$ we have that f is either monotonically strictly increasing or monotonically strictly decreasing. So f is injective and hence invertible.

We can now show that J is open. If $y \in J$ we put $x = f^{-1}(y) \in I$. As I is open we can find $x_1, x_2 \in I$ such that $x_1 < x < x_2$ and then we have that $y = f(x$ is between $f(x_1)$ and $f(x_2)$. So y cannot be an endpoint, i.e., J is open.

We now consider a $y \in J$. Before we find the derivative of the inverse we need a little bit of preparation. We put $x = f^{-1}(y)$ and choose $a, b \in I$ such that $a < x < b$. Now $f([a, b])$ is a closed interval $[c, d] \subseteq J$ and $c < y < d$. If $h > 0$ and $y + h \in [c, d]$ then we put $k = f^{-1}(y + h) - f^{-1}(y)$, i.e., $y + h = f(x + k)$. By the mean value theorem we have

$$h = y + h - y = f(x + k) - f(x) = f'(\xi)k$$

and hence

$$|k| = \left|\frac{h}{f'(\xi)}\right| \leq \frac{|h|}{\min_{t\in[a,b]} |f'(t)|}.$$

We see that $h \to 0 \Rightarrow k \to 0$ and hence

$$\frac{f^{-1}(y+h)-f^{-1}(y)}{h} = \frac{x+k-x}{f(x+k)-f(x)}$$
$$= \frac{1}{\frac{f(x+k)-f(x)}{k}} \xrightarrow[h\to 0]{} \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}\,. \qquad \square$$

Example 2.18. The elementary functions are differentiable with derivatives given in the following table

$f(x)$	$\log x$	$\exp x$	$\cos x$	$\sin x$
$f'(x)$	$\frac{1}{x}$	$\exp x$	$-\sin x$	$\cos x$

We can define the natural logarithm as $\log(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t$ so the derivative is clearly $\frac{1}{x}$. We can then define exp as the inverse of log and then we have that the derivative is $\frac{1}{\frac{1}{\exp x}} = \exp x$. The derivatives of cosine and sine are derived in Appendix C.

Given this we can find the derivatives of more functions

Example 2.19. If $a > 0$ then the following functions are differentiable

$f(x)$	x^α	a^x	$\tan x = \frac{\sin x}{\cos x}$		$\cot x = \frac{\cos x}{\sin x}$	
$f'(x)$	$\alpha x^{\alpha-1}$	$\log(a)a^x$	$\frac{1}{\cos^2 x} = 1+\tan^2 x$		$\frac{-1}{\sin^2 x} = -(1+\cot^2 x)$	
$f(x)$	$\log_a(x)$	$\arccos x$	$\arcsin x$	$\arctan x$	$\operatorname{arccot} x$	
$f'(x)$	$\frac{1}{x\log a}$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$\frac{-1}{1+x^2}$	

2.3 The Riemann integral

Given a continuous function $f : [a,b] \to \mathbb{R}$ we want to make sense of the area under the graph, or rather the area between the x-axis and the graph, where the portion over the x-axis is counted positive and the area below the x-axis is counted negative, see Figure 2.5. In order to that we partition the interval $[a,b]$ in n pieces, i.e., we pick $x_k \in [a,b]$ for $k = 0,1,\dots,n$ such that $a = x_0 < x_1 < \cdots < x_n = b$ and consider the closed intervals $[x_{k-1}, x_k]$ for $k = 1,2,\dots,n$. In each interval the function f attains its minimum $c_k = \min_{x\in[x_{k-1},x_k]} f(x)$ and its maximum $d_k = \max_{x\in[x_{k-1},x_k]} f(x)$. We can now write down a *lower sum* and an *upper sum*

$$L((x_k)_{k=0}^n) = \sum_{k=1}^n c_k(x_k - x_{k-1})\,, \quad U((x_k)_{k=0}^n) = \sum_{k=1}^n d_k(x_k - x_{k-1})\,, \tag{2.19}$$

respectively, see Figure 2.6.

A partition $a = x'_0 < x'_1 < \cdots < x'_{n'} = b$ is called a *refinement* of the part-ion $a = x_0 < x_1 < \cdots < x_n = b$ if it is obtained by partitioning some (or all) of the intervals $[x_{k-1}, x_k]$, i.e., if $\{x_k \mid k = 0,\dots,n\} \subseteq \{x'_k \mid k = 0,\dots,n'\}$. If we refine a part-ion then the corresponding lower sum becomes bigger and the corresponding upper sum becomes smaller:

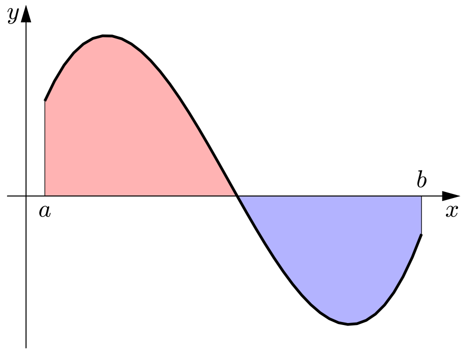

Figure 2.5: The area over the x-axis is counted positive (pink). The area under the x-axis is counted negative (light blue).

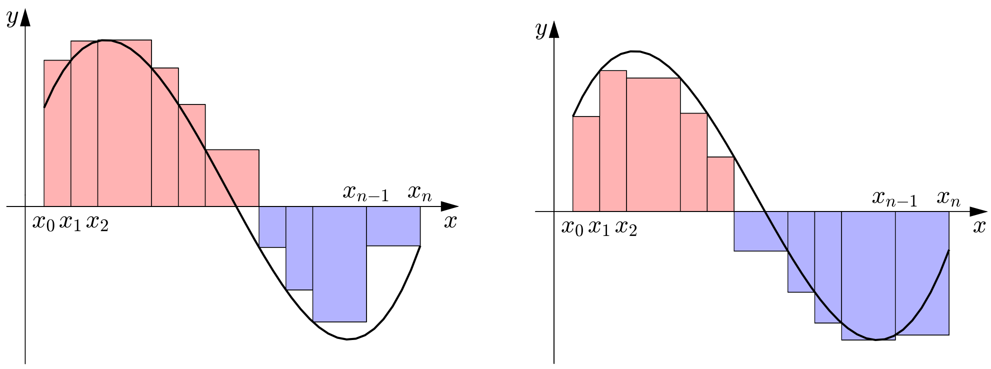

Figure 2.6: To the left the signed area of an upper sum. To the right of a lower sum.

Lemma 2.41. *If $a = x'_0 < x'_1 < \cdots < x'_{n'} = b$ is a refinement of $a = x_0 < x_1 < \cdots < x_n = b$ then $L((x_k)_{k=0}^n) \leq L((x'_k)_{k=0}^{n'}) \leq U((x'_k)_{k=0}^{n'}) \leq U((x_k)_{k=0}^n)$.*

Proof. As $c'_k = \min_{x\in[x'_{k-1},x'_k]} f(x) \leq \max_{x\in[x_{k-1},x_k]} f(x) = d'_k$ we clearly have $L((x'_k)_{k=0}^{n'}) \leq U((x'_k)_{k=0}^{n'})$.

If $a \leq t_1 < t_2 < t_3 \leq b$ then $\min_{x\in[t_1,t_3]} f(x)$ is a lower bound for f on both $[t_1, t_2]$and $[t_2, t_3]$. Hence

$$\begin{aligned}\min_{x\in[t_1,t_3]} f(x)(t_3 - t_1) &= \min_{x\in[t_1,t_3]} f(x)((t_2 - t_1) + (t_3 - t_2)) \\ &\leq \min_{x\in[t_1,t_2]} f(x)(t_2 - t_1) + \min_{x\in[t_1,t_3]} f(x)(t_3 - t_2)\,.\end{aligned}$$

So $L((x_k)_{k=0}^n) \leq L((x'_k)_{k=0}^{n'})$. Likewise, $\max_{x\in[t_1,t_3]} f(x)$ is an upper bound for f on both $[t_1, t_2]$and $[t_2, t_3]$ and the inequality $U((x'_k)_{k=0}^{n'}) \leq U((x_k)_{k=0}^n)$ follows. □

Any lower sum is smaller than any upper sum

Lemma 2.42. *If $a = x_0 < x_1 < \cdots < x_n = b$ and $a = x'_0 < x'_1 < \cdots < x'_{n'} = b$ are two partitions of $[a, b]$ then $L((x'_k)_{k=0}^{n'}) \leq U((x_k)_{k=0}^n)$.*

Proof. By sorting the set $\{x_0, x_1, \ldots , x_n, x'_0, x'_1, \ldots , x'_{n'}\}$ we obtain a partition $a = x''_0 < x''_1 < \cdots < x''_{n''} = b$ that is a refinement of both of the given partitions. Now Lemma 2.41 tells us that $L((x'_k)_{k=0}^{n'}) \leq L((x''_k)_{k=0}^{n''}) \leq U((x''_k)_{k=0}^{n''}) \leq U((x_k)_{k=0}^n)$. □

So the set of all lower sums are bounded from above (by any upper sum) and set of all upper sums are bounded from below (by any lower sum). Hence the supremum of the former and the infimum of the latter exist. It turns out that these two numbers are equal:

Theorem 2.43. *Let*

$$\mathcal{L} = \{L((x_k)_{k=0}^n) \mid a \leq x_0 < x_1 < \cdots < x_n = b\}\,,$$

and

$$\mathcal{U} = \{U((x_k)_{k=0}^n) \mid a \leq x_0 < x_1 < \cdots < x_n = b\}\,,$$

be the sets of all lower and upper sums, respectively. Then $\sup \mathcal{L} = \inf \mathcal{U}$.

Proof. If $L \in \mathcal{L}$ and $U \in \mathcal{U}$ then $L \leq U$ so we clearly have $\sup \mathcal{L} \leq \inf \mathcal{U}$.

Let $\epsilon > 0$ be given. As f is continuous Theorem 2.12 says that f is uniformly continuous. So there exist $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, for all $x, y \in [a, b]$.

We now choose $n \in \mathbb{N}$ such that $\frac{b-a}{n} < \delta$ and we put $x_k = \frac{n-k}{n}a + \frac{k}{n}b$ for $k = 0, 1, \ldots , n$. This gives a partition $a = x_0 < x_1 < \cdots < x_n = b$ where $x_k - x_{k-1} = \frac{b-a}{n} < \delta$ for all $k = 1, 2, \ldots , n$.

Letting $c_k = \min_{x\in[x_{k-1},x_k]} f(x)$ and $d_k = \max_{x\in[x_{k-1},x_k]} f(x)$ we have $c_k = f(s_k)$ and $d_k = f(t_k)$ for some $s_k, t_k \in [x_{k-1}, x_k]$.

Then $|t_k - s_k| \le x_k - x_{k-1} < \delta$ and $d_k - c_k < \epsilon$ for all $k = 1, 2, \dots, n$. We finally have

$$U((x_k)_{k=0}^n) - L((x_k)_{k=0}^n) = \sum_{k=1}^n d_k(x_k - x_{k-1}) - \sum_{k=1}^n c_k(x_k - x_{k-1})$$
$$= \sum_{k=1}^n (d_k - c_k)\frac{b-a}{n} < \sum_{k=1}^n \epsilon\frac{b-a}{n} = \epsilon\,.$$

As $L((x_k)_{k=0}^n) \le \sup\mathcal{L} \le \inf\mathcal{U} \le U((x_k)_{k=0}^n)$ we have

$$\inf\mathcal{U} - \sup\mathcal{L} \le U((x_k)_{k=0}^n) - L((x_k)_{k=0}^n) < \epsilon\,.$$

Hence $\inf\mathcal{U} < \sup\mathcal{L} + \epsilon$ for all $\epsilon > 0$ and that implies $\inf\mathcal{U} \le \sup\mathcal{L} \le \inf\mathcal{U}$, i.e., $\sup\mathcal{L} = \inf\mathcal{U}$. □

We now define the integral of f over $[a, b]$ as this common value:

Definition 2.44. Let $f : [a, b] \to \mathbb{R}$ be continuous. The *integral* of f is

$$\int_a^b f(x)\,\mathrm{d}x = \sup\mathcal{L} = \inf\mathcal{U}\,,$$

where $\mathcal{L}$ and $\mathcal{U}$ are the the sets of lower and upper sums, respectively.

If we examine the proof of Theorem 2.43 we see that we have

Lemma 2.45. *Let $f : [a, b] \to \mathbb{R}$ be continuous and let $\epsilon > 0$ be given. If $\delta > 0$, $|x - y| < \delta \Rightarrow |f(x) - f(y) < \epsilon$, and $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a, b]$ such that $x_k - x_{k-1} < \delta$, then*

$$0 \le U((x_k)_{k=0}^n) - \int_a^b f(x)\,\mathrm{d}x < \epsilon\,,$$
$$0 \le \int_a^b f(x)\,\mathrm{d}x - L((x_k)_{k=0}^n) < \epsilon\,.$$

So in principle we can approximate the integral $\int_a^b f(x)\,\mathrm{d}x$ by a lower or upper sum of a sufficiently dense partition of $[a, b]$, but then we will have to find the minimum or maximum of f on all the subintervals $[x_{k-1}, x_k]$. That is clearly very cumbersome. Instead we can just evaluate f in any point $\xi_k \in [x_{k-1}, x_k]$ and thereby obtain what is called a *Riemann sum*

$$M((x_k)_{k=0}^n, (\xi_k)_{k=1}^n) = \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1})\,., \tag{2.20}$$

see Figure 2.7. In the figure we have used the mid points $\xi_k = \frac{x_{k-1}+x_k}{2}$, but that is not important.

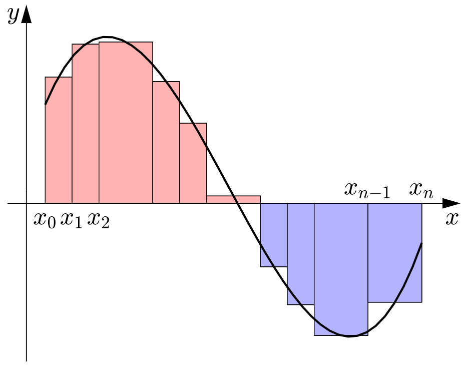

Figure 2.7: The signed area of a Riemann sum.

Remark 2.46. Observe that both the lover sum $L((x_k)_{k=0}^n)$ and the upper sum $U((x_k)_{k=0}^n)$ is a Riemann sum and that

$$L((x_k)_{k=0}^n) \leq M((x_k)_{k=0}^n, (\xi_k)_{k=1}^n) \leq U((x_k)_{k=0}^n)$$

for any Riemann sum $M((x_k)_{k=0}^n, (\xi_k)_{k=1}^n)$.

Theorem 2.47. *Let $f : [a, b] \to \mathbb{R}$ be continuous and let $\epsilon > 0$ be given. If $\delta > 0$, $|x - y| < \delta \Rightarrow |f(x) - f(y) < \epsilon$, $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a, b]$ such that $x_k - x_{k-1} < \delta$ and $\xi_k \in [x_{k-1}, x_k]$, then*

$$\left| \int_a^b f(x)\, \mathrm{d}x - \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) \right| < \epsilon\,.$$

Proof. As $L((x_k)_{k=0}^n) \leq \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) \leq U((x_k)_{k=0}^n)$ the result is a consequence of Lemma 2.45. □

Remark 2.48. The theorem says that the integral is the limit of Riemann sums, where the largest difference in the partitions ($\max(x_k - x_{k-1})$) goes to zero. That means that many properties of Riemann sums are valid for integrals

As a Riemann sum is linear in f, so is the integral:

Theorem 2.49. *If $f, g : [a, b] \to \mathbb{R}$ are two continuous functions and $\lambda \in \mathbb{R}$ then*

$$\int_a^b (f(x) + g(x))\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x = \int_a^b g(x)\, \mathrm{d}x\,,$$
$$\int_a^b \lambda f(x)\, \mathrm{d}x = \lambda \int_a^b f(x)\, \mathrm{d}x\,.$$

Proof. Left as Exercise 2.8. □

For a Riemann sum we have $|\sum_{k=1}^n f(\xi)(x_k - x_{k-1})| \leq \sum_{k=1}^n |f(\xi)|(x_k - x_{k-1})$. Similar for an integral:

Theorem 2.50. *If $f : [a,b] \to \mathbb{R}$ is continuous then*

$$\left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f(x)|\,\mathrm{d}x\,.$$

Proof. Left as Exercise 2.9. □

We can split an integral in two (or more) integrals:

Theorem 2.51. *Let $f : [a,c] \to \mathbb{R}$ be continuous and let $b \in [a,c]$ then*

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x\,.$$

Proof. Let $a = x_0 < x_1 < \cdots < x_n = c$ be a partition of $[a,c]$. Either we have $b = x_m$ for some m or we have a m such that $x_{m-1} < b < x_m$. In the first case we put $x'_k = x_k$ all $k = 0,1,\ldots,n$ and in the second case we put $x'_k = x_k$ for $k = 0,1,\ldots,m-1$, $x'_m = b$, and $x'_k = x_{k-1}$ for $k = m+1,1,\ldots,n+1$. In both cases we have a refinement $a = x'_0 < x'_1 < \cdots < x'_{n'} = c$ where $b = x'_m$.

We now have

$$L((x'_k)_{k=0}^m) \leq \int_a^b f(x)\,\mathrm{d}x \leq U((x'_k)_{k=0}^m)\,,$$
$$L((x'_k)_{k=m}^{n'}) \leq \int_b^c f(x)\,\mathrm{d}x \leq U((x'_k)_{k=m}^{n'})\,,$$

and hence

$$\begin{aligned} L((x_k)_{k=0}^n) \leq L((x'_k)_{k=0}^{n'}) &= L((x'_k)_{k=0}^m) + L((x'_k)_{k=m}^{n'}) \\ &\leq \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x \\ &\leq U((x'_k)_{k=0}^m) + U((x'_k)_{k=m}^{n'}) = U((x'_k)_{k=0}^{n'}) \leq U((x_k)_{k=0}^n) \end{aligned}$$

As the partition was arbitrary this implies that

$$\int_a^c f(x)\,\mathrm{d}x = \sup \mathcal{L} \leq \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x \leq \inf \mathcal{U} = \int_a^c f(x)\,\mathrm{d}x\,.$$ □

The integral of a positive function is positive.

Theorem 2.52. *If $f : [a,b] \to \mathbb{R}$ is continuous and $f(x) \geq 0$ for all $x \in [a,b]$ then $\int_a^b f(x)\,\mathrm{d}x \geq 0$. Furthermore, $\int_a^b f(x)\,\mathrm{d}x = 0$ if and only if $f(x) = 0$ for all $x \in [a,b]$.*

Proof. If $f(x) \geq 0$ for all $x \in [a,b]$ then all Riemann sums $\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \geq 0$ and as $\int_a^b f(x)\,\mathrm{d}x$ is the limit of Riemann sums it is non negative too.

Suppose $f(x_0) \neq 0$ for a point $x_0 \in [a,b]$, then $f(x_0) > 0$. As f is continuous we can find $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \frac{f(x_0)}{2}$. Then $|x - x_0| < \delta \Rightarrow f(x) \geq \frac{f(x_0)}{2}$. Hence we can can find $a \leq c < d \leq b$ such that $f(x) \geq \frac{f(x_0)}{2}$ for all $x \in [c,d]$. As $f(x) - \frac{f(x_0)}{2} \geq 0$ for all $x \in [c,d]$ we have $\int_c^d \left(f(x) - \frac{f(x_0)}{2}\right) \mathrm{d}x \geq 0$ and then

$$\begin{aligned}\int_c^d f(x)\,\mathrm{d}x &= \int_c^d \left(f(x) - \frac{f(x_0)}{2} + \frac{f(x_0)}{2}\right) \mathrm{d}x \\ &= \int_c^d \left(f(x) - \frac{f(x_0)}{2}\right) \mathrm{d}x + \int_c^d \frac{f(x_0)}{2}\,\mathrm{d}x \\ &= \int_c^d \left(f(x) - \frac{f(x_0)}{2}\right) \mathrm{d}x + \frac{f(x_0)}{2}(d-c) \geq \frac{f(x_0)}{2}(d-c)\,.\end{aligned}$$

We now have

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^d f(x)\,\mathrm{d}x + \int_d^b f(x)\,\mathrm{d}x \geq \int_c^d f(x)\,\mathrm{d}x > 0\,. \qquad \square$$

Integration preserves the ordering

Theorem 2.53. *If $f, g : [a,b] \to \mathbb{R}$ are continuous and $f(x) \geq g(x)$ for all $x \in [a,b]$ then $\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b y(x)\,\mathrm{d}x$. Furthermore, $\int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x$ if and only if $f(x) = g(x)$ for all $x \in [a,b]$.*

Proof. Consider the function $f - g$ and use Theorem 2.52. $\square$

Remark 2.54. The lower limit is allowed to be larger than the upper limit. If $f : [a,b] \to \mathbb{R}$ is continuous then we put $\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$.

We have a mean value theorem for integration:

Theorem 2.55 (Mean value theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous then there exist $\xi \in]a,b[$ such that $\int_a^b f(x)\,\mathrm{d}x = f(\xi)(b-a)$.*

Proof. If f is constant the theorem is true for any $\xi \in]a,b[$.

Otherwise Theorem 2.9 tells us that $f([a,b]) = [c,d]$ and $c < d$. Then $c(b-1) < \int_a^b f(x)\,\mathrm{d}x < d(b-a)$. So there exist $y \in]c,d[$ such that $\int_a^b f(x)\,\mathrm{d}x = y(b-a)$.

We can now find $x_1, x_2 \in [a,b]$ such that $f(x_1) = c$ and $f(x_2) = d$ and then we can find ξ between x_1 and x_2 with $f(\xi) = y$. As $c < y < d$ we must have $\xi \neq x_1$ and $\xi \neq x_2$ so $\xi \in]a,b[$. $\square$

Remark 2.56. It is easy to see that we can find $\xi \in [a,b]$ such that $\int_a^b f(x)\,\mathrm{d}x = f(\xi)(b-a)$. The hard part is to show that we can avoid a and b.

The integral is a differentiable function of its upper (and lower) limit.

Theorem 2.57. *Let $f : [a, b] \to \mathbb{R}$ be continuous.*

The function $F : [a, b] \to \mathbb{R}$ given by $F(t) = \int_a^t f(x)\,\mathrm{d}x$ is differentiable with derivative $F'(t) = f(t)$.

Proof. $F(t+h) - F(t) = \int_a^{t+h} f(x)\,\mathrm{d}x - \int_a^t f(x)\,\mathrm{d}x = \int_t^{t+h} f(x)\,\mathrm{d}x$. Given $\epsilon > 0$ choose $\delta > 0$ such that $|t - s| < \delta \Rightarrow |f(t) - f(s)| < \epsilon$. The mean value theorem above now give us a$\xi \in [t, t+h]$ such that $\int_t^{t+h} f(x)\,\mathrm{d}x = f(\xi)h$. We now have $\left|\frac{F(t+h)-F(t)}{h} - f(t)\right| = |f(\xi) - f(t)| < \epsilon$ for $|h| < \delta$. □

Remark 2.58. We have

$$\int_{t+h}^{b} f(x)\,\mathrm{d}x - \int_{t}^{b} f(x)\,\mathrm{d}x = \int_{t+h}^{t} f(x)\,\mathrm{d}x = -\int_{t}^{t+h} f(x)\,\mathrm{d}x\,.$$

So $\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b f(x)\,\mathrm{d}x = -f(t)$.

Definition 2.59. An *anti derivative* to a function $f :]a, b[\to \mathbb{R}$ is a function $F :]a, b[\to \mathbb{R}$ such that $F'(x) = f(x)$ for all $x \in]a, b[$.

Lemma 2.60. *If $F, G :]a, b[\to \mathbb{R}$ are differentiable and $F'(x) = G'(x)$ for all $x \in]a, b[$ then $F - G$ is constant.*

Proof. Let $x, y \in]a, b[$ with $x < y$. As $H = F - G$ is differentiable with derivative $H'(x) = F'(x) - G'(x) = 0$ the mean value theorem (Theorem 2.27) gives us $\xi \in]x, y[$ such that $H(y) - H(x) = H'(\xi)(y - x) = H'(\xi)(y - x) = 0$. That is, $H(y) = H(x)$. □

Theorem 2.61 (Fundamental theorem of calculus)**.** *If $f :]a, b[\to \mathbb{R}$ is continuous and $x_0 \in]a, b[$ then all anti derivatives to f is given by*

$$F(x) = \int_{x_0}^{x} f(x)\,\mathrm{d}x + k\,.$$

where $k \in \mathbb{R}$ is an arbitrary constant.

Proof. By Theorem 2.57 such functions F are indeed anti derivatives and Lemma 2.60 tells that they are the only ones. □

Conversely we have

Theorem 2.62. *If $[a, b] \subseteq]c, d[$, $f : [a, b] \to \mathbb{R}$ is continuous, $F :]c, d[\to \mathbb{R}$ is differentiable and $F' = f$ then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.*

Proof. If we for $x \in [a, b]$ let $G(x) = \int_a^x f(x)\,\mathrm{d}x$ then $G(x) = F(x) + k$ for $x \in]a, b[$. As $G(a) = 0$ we have $\int_a^x f(x)\,\mathrm{d}x = G(x) - G(a) = F(x) - F(a)$ for all $x \in]a, b[$ and by continuity we have $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$. □

Theorem 2.63. *If* $[a, b] \subseteq]c, d[$, $g :]c, d[\to [a', b']$ *is differentiable, and* $f : [a', b'] \to \mathbb{R}$ *is continuous then*

$$\int_a^b f(g(t))g'(t)\,\mathrm{d}t = \int_{g(a)}^{g(b)} f(x)\,\mathrm{d}x\,.$$

Proof. If F is an anti derivative of f then $\frac{\mathrm{d}}{\mathrm{d}t}F(g(t)) = F'(g(t))g'(t) = f(g(t))g'(t)$, i.e., $F \circ g$ is an anti derivative of $(f \circ g) \cdot g'$. Thus

$$\int_a^b f(g(t))g'(t)\,\mathrm{d}t = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(x)\,\mathrm{d}x\,. \qquad \square$$

Remark 2.64. Formally we substitute $x = g(t)$ and say $\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = g'(t)\,\mathrm{d}t$.

2.4 Exercises

Exercise 2.1. Prove Theorem 2.5

Exercise 2.2. Prove Theorem 2.6

Exercise 2.3. Prove Theorem 2.19

Exercise 2.4. Prove Theorem 2.20

Exercise 2.5. Find the 2nd degree Taylor polynomial of $\cos x$ and $\sin x$ at 0.

Exercise 2.6. Find the 2nd degree Taylor polynomial of $\exp x$ at 0.

Exercise 2.7. Find the 2nd degree Taylor polynomial of $\log x$ at 1 .

Exercise 2.8. Prove Theorem 2.49, i.e., integration is linear in the integrand.

Exercise 2.9. Prove Theorem 2.50, i.e., $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x\,.$

Exercise 2.10. The hyperbolic cosine and sine are defined by $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and $\sinh(x) = \frac{e^x - e^{-x}}{2}$, respectively. Find their derivatives.

Chapter 3

Functions of several variables

3.1 Introduction

We will use the euclidean norm to measure distance in $\mathbb{R}^n$, i.e., if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ are two points in $\mathbb{R}^n$ then the distance between them is

$$\|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2}\,. \tag{3.1}$$

Let $A \subseteq \mathbb{R}^n$, a vector function $A \to \mathbb{R}^m$ is of the form

$$(x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))\,,$$

where $f_1, \ldots, f_n$ are real functions $A \to \mathbb{R}$. We can write it more compact as $\boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{f} = (f_1, \ldots, f_m)$.

Example 3.1. A *quadratic form* is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c\,, \tag{3.2}$$

where $c \in \mathbb{R}$, $\boldsymbol{b} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, and we consider elements $\boldsymbol{x} \in \mathbb{R}^n$ as column vectors. As $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}$ is 1×1 we have $\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} = (\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x})^T = \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{x}$ and hence

$$f(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{x}^T (\boldsymbol{A} + \boldsymbol{A}^T) \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c\,.$$

As $\frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$ is symmetric we can always assume that $\boldsymbol{A}$ is symmetric.

Example 3.2. If $\boldsymbol{A}$ is symmetric all eigenvalues λ_k are real and we can find an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ for $\mathbb{R}^n$ consisting of eigenvectors for $\boldsymbol{A}$. If $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + \cdots + x_n \boldsymbol{e}_n$ and $\boldsymbol{b} = b_1 \boldsymbol{e}_1 + \cdots + b_n \boldsymbol{e}_n$ then we can write (3.2) as

$$f(\boldsymbol{x}) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 + b_1 x_1 + \cdots + b_n x_n + c\,.$$

If $\lambda_k \neq 0$ then $\lambda_k x_k^2 + b_k x_k = \lambda_k \left(x_k + \frac{b_k}{2\lambda_k} \right)^2 - \frac{b_k^2}{4\lambda_k}$. So by sorting the eigenvalues properly we can write

$$f(\boldsymbol{x}) = \lambda_1 \left(x_1 - \frac{b_1^2}{4\lambda_1} \right)^2 + \cdots + \lambda_m \left(x_m - \frac{b_m^2}{4\lambda_m} \right)^2 + b_{m+1} x_{m+1} + \cdots + b_n x_n + C\,, \tag{3.3}$$

where $\lambda_1, \ldots, \lambda_m \neq 0$, $\lambda_{m+1}, \ldots, \lambda_n = 0$, and $C = c - \frac{b_1^2}{4\lambda_1} - \cdots - \frac{b_m^2}{4\lambda_m}$.

3.1.1 Quadratic forms in the plane

We consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ of the form (3.2). We are interested in the *level sets* of f, i.e., the solutions to the equation

$$f(\boldsymbol{x}) = \text{constant}\,. \tag{3.4}$$

If $\boldsymbol{A} = \boldsymbol{0}$ we simply have a linear equation $\boldsymbol{b}^T\boldsymbol{x} = \text{constant}$. So we will assume $\boldsymbol{A} \neq \boldsymbol{0}$.

As we saw in Example 3.1 we may assume that $\boldsymbol{A}$ is symmetric. Then the eigenvalues are real and we can find an orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ consisting of eigenvectors of $\boldsymbol{A}$ with eigenvalues λ_1 and λ_2, respectively. At least one of the eigenvalues are non zero and we may assume it is λ_1 (otherwise exchange $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$). We may also assume $\lambda_1 > 0$, otherwise multiply the equation (3.4) by -1 on both sides. There is a number of different cases:

$\lambda_1 > 0,\ \lambda_2 > 0$: Now $\boldsymbol{A}$ is regular and if we put $\boldsymbol{x}_0 = -\frac{1}{2}\boldsymbol{A}^{-1}\boldsymbol{b}$, then a direct calculation shows that

$$f(\boldsymbol{x}) = (\boldsymbol{x} - \boldsymbol{x}_0)^T\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}_0) + c - \boldsymbol{x}_0^T\boldsymbol{A}\boldsymbol{x}_0\,.$$

We can now write $\boldsymbol{x} = \boldsymbol{x}_0 + x\boldsymbol{e}_1 + y\boldsymbol{e}_2$ and then

$$f(\boldsymbol{x}) = \lambda_1 x^2 + \lambda_2 y^2 + c - \boldsymbol{x}_0^T\boldsymbol{A}\boldsymbol{x}_0\,.$$

By subtracting $c - \boldsymbol{x}_0^T\boldsymbol{A}\boldsymbol{x}_0$ from both sides of (3.4) we end up with an equation of the form

$$\lambda_1 x^2 + \lambda_2 y^2 = \text{constant}\,. \tag{3.5}$$

If the constant is negative we have no solutions. If the constant is zero the solution is a single point $\boldsymbol{x} = \boldsymbol{x}_0$, and if the constant is positive we have an *ellipse*, see Figure 3.1 left.

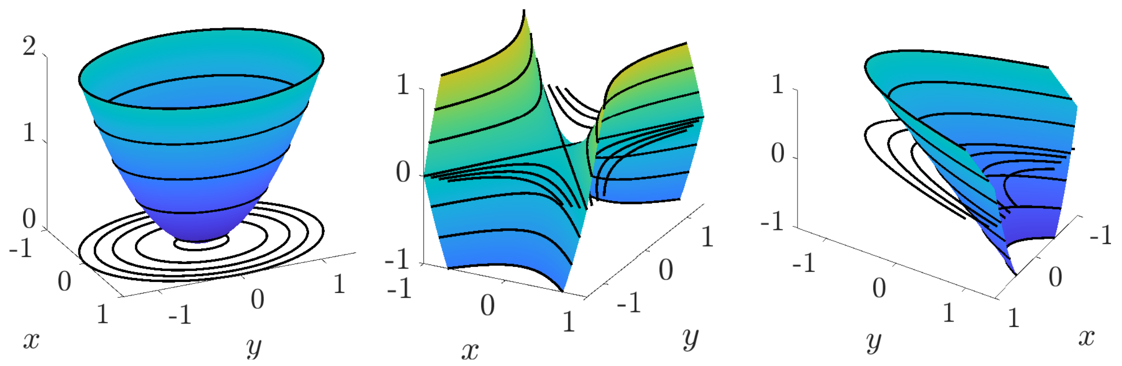

Figure 3.1: Left to right: The case $\lambda_1, \lambda_2 > 0$, the case $\lambda_1 > 0,\ \lambda_2 < 0$, the case $\lambda_1 > 0,\ \lambda_2 = 0,\ \langle \boldsymbol{b}, \boldsymbol{e}_2 \rangle \neq 0$

$\lambda_1 > 0,\ \lambda_2 < 0$: The matrix $\boldsymbol{A}$ is still regular and we can repeat the previous argument. So we still end up with the equation (3.5). But, now we have solutions, a pair of *hyperbolas*, for all values of the constant, see Figure 3.1 Riemann.

$\lambda_1 > 0,\ \lambda_2 = 0,\ \langle \boldsymbol{e}_2, \boldsymbol{b}\rangle \neq 0$: We can write $\boldsymbol{b} = a\boldsymbol{e}_1 + b\boldsymbol{e}_2$ where $b \neq 0$. We may assume that $b < 0$ otherwise replace $\boldsymbol{e}_2$ with $-\boldsymbol{e}_2$. By multiplying the equation (3.4) by $\frac{1}{|b|}$ on both sides we may furthermore assume that $b = -1$. If we write $\boldsymbol{x} = \left(x + \frac{a}{2\lambda_1}\right)\boldsymbol{e}_1 + y\boldsymbol{e}_2$ then we have

$$f(\boldsymbol{x}) = \lambda_1 x^2 - y + c - \frac{a^2}{4\lambda_1}\,.$$

By subtracting $c - \frac{a^2}{4\lambda_1}$ from both sides of (3.4) we end up with an equation of the form

$$\lambda_1 x^2 - y = \text{constant}\,.$$

The solutions are *parabolas*, $y = \lambda_1 x^2 - \text{constant}$, see Figure 3.1 right.

$\lambda_1 > 0,\ \lambda_2 = 0,\ \langle \boldsymbol{e}_2, \boldsymbol{b}\rangle = 0$: Now $\boldsymbol{b} = a\boldsymbol{e}_1$ and if we write $\boldsymbol{x} = \left(x + \frac{a}{2\lambda_1}\right)\boldsymbol{e}_1 + y\boldsymbol{e}_2$ then we have $f(\boldsymbol{x}) = \lambda_1 x^2 + c - \frac{a^2}{4\lambda_1}$. By subtracting $c - \frac{a^2}{4\lambda_1}$ from both sides of (3.4) we end up with an equation of the form

$$\lambda_1 x^2 = \text{constant}\,.$$

If the constant is negative there are no solutions. If the constant is zero the solution is a line given by $x = 0$ and if the constant is negative the solution is two parallel lines given by $x = \pm\sqrt{\frac{\text{constant}}{\lambda_1}}$.

3.2 Continuity

The definition of continuity for vector functions of several variables is the same as for functions of one variable

Definition 3.1. Let $A \subseteq \mathbb{R}^n$ and $\boldsymbol{x}_0 \in A$. A vector function $\boldsymbol{f} : A \to \mathbb{R}^m$ is called *continuous at a point* $\boldsymbol{x}_0$ if we for all positive numbers ϵ can find a positive number δ such that if $\boldsymbol{x} \in A$ and $\|\boldsymbol{x} - \boldsymbol{x}_0| < \delta$ then $\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| < \epsilon$. With logical symbols this can be written

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall \boldsymbol{x} \in A : \|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta \implies \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| < \epsilon\,. \tag{3.6}$$

If $\boldsymbol{f}$ is continuous at all point $\boldsymbol{x} \in A$ then $\boldsymbol{f}$ is called *continuous*. With logical symbols this can be written

$$\forall \boldsymbol{y} \in A\, \forall \epsilon > 0\, \exists \delta > 0\, \forall \boldsymbol{x} \in A : \|\boldsymbol{x} - \boldsymbol{y}\| < \delta \implies \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| < \epsilon\,. \tag{3.7}$$

Remark 3.2. The continuity condition is often written as

$$\boldsymbol{f}(\boldsymbol{x}) \to \boldsymbol{f}(\boldsymbol{x}_0) \text{ for } \boldsymbol{x} \to \boldsymbol{x}_0 \,. \tag{3.8}$$

Example 3.3. Let $\boldsymbol{A}$ be an $m \times n$ matrix. The linear function $L : \mathbb{R}^n \to \mathbb{R}^m$ given by $L(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ is continuous in all points $\boldsymbol{x}_0 \in \mathbb{R}^n$. Indeed, if $\boldsymbol{A} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}$ then we let $c = \max\{|a_{k,\ell}|\}$. If $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ we have

$$|y_k| = |a_{k,1}x_1 + \cdots + a_{k,n}x_n| \le |a_{k,1}||x_1| + \cdots + |a_{k,n}||x_n| \le cn\|\boldsymbol{x}\| \,,$$

and hence

$$\|\boldsymbol{y}\| = \sqrt{|y_1|^2 + \cdots + |y_m|^2} \le \sqrt{m(cn\|\boldsymbol{x}\|)^2} = \sqrt{m}nc\|\boldsymbol{x}\| \,.$$

So if $\epsilon > 0$, $0 < \delta \le \frac{\epsilon}{\sqrt{m}nc}$, and $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ then

$$\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{x}_0\| = \|\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}_0)\| \le \sqrt{m}nc\|\boldsymbol{x} - \boldsymbol{x}_0\| < \sqrt{m}nc\delta \le \epsilon \,.$$

Example 3.4. A quadratic function (3.2) is continuous. We will later see that a differentiable function is continuous (Theorem 3.17) and a quadratic function is differentiable (Example 3.6). So we will not show it now.

The following four lemmas give other examples of continuous functions

Lemma 3.3 (Projection on the kth coordinate)**.** *Let $k \in \{1, 2, \ldots, n\}$. The function $P_k : \mathbb{R}^n \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto x_k$ is continuous at all points $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.*

Proof. We have

$$\begin{aligned} |P_k(\boldsymbol{x}) - P_k(\boldsymbol{y})| = |x_k - y_k| &= \sqrt{(x_k - y_k)^2} \\ &\le \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \|\boldsymbol{x} - \boldsymbol{y}\| \,. \end{aligned}$$

If $\epsilon > 0$ and $0 < \delta \le \epsilon$ then $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta \Rightarrow |P_k(\boldsymbol{x}) - P_k(\boldsymbol{x}_0)| < \epsilon$. □

Lemma 3.4 (Diagonal)**.** *The vector function* diag $: \mathbb{R}^n \to \mathbb{R}^{2n} : \boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{x})$ *is continuous at all points $\boldsymbol{x} \in \mathbb{R}^n$.*

Proof. Given $\boldsymbol{x}_0 \in \mathbb{R}^n$ and $\epsilon > 0$ we put $\delta = \frac{1}{2}\epsilon$. If $\boldsymbol{x} \in \mathbb{R}^n$ and $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ then

$$\begin{aligned} \|(\boldsymbol{x}, \boldsymbol{x}) - (\boldsymbol{x}_0, \boldsymbol{x}_0)\| &= \|(\boldsymbol{x} - \boldsymbol{x}_0, \boldsymbol{x} - \boldsymbol{x}_0)\| \\ &= \sqrt{\|\boldsymbol{x} - \boldsymbol{x}_0\|^2 + \|\boldsymbol{x} - \boldsymbol{x}_0\|^2} < \sqrt{\delta^2 + \delta^2} = \sqrt{2}\delta = \frac{\sqrt{2}}{2}\epsilon < \epsilon \,. \end{aligned}$$ □

Lemma 3.5. *Addition* add $: \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} + \boldsymbol{y}$ *is continuous at all points $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n}$.*

Proof. Given $(\boldsymbol{x}_0, \boldsymbol{y}_0) \in \mathbb{R}^{2n}$ and $\epsilon > 0$ we put $\delta = \frac{1}{2}\epsilon$. If $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n}$ and $\|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}_0, \boldsymbol{y}_0)\| < \delta$ then $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta$ and $\|\boldsymbol{y} - \boldsymbol{y}_0\| < \delta$. Hence

$$\begin{aligned}\|\boldsymbol{x} + \boldsymbol{y} - (\boldsymbol{x}_0 + \boldsymbol{y}_0)\| &= \|\boldsymbol{x} - \boldsymbol{x}_0 + \boldsymbol{y} - \boldsymbol{y}_0\| \\ &\le \|\boldsymbol{x} - \boldsymbol{x}_0\| + \|\boldsymbol{y} - \boldsymbol{y}_0\| < \delta + \delta = \epsilon\,. \qquad \square\end{aligned}$$

Lemma 3.6. *Multiplication* mult $: \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto xy$ *is continuous at all points* $(x, y) \in \mathbb{R}^2$.

Proof. Given $\boldsymbol{x}_0 = (x_0, y_0)$ and $\epsilon > 0$. Put $\delta = \min\left\{1, \frac{\epsilon}{2+2|x_0|}, \frac{\epsilon}{2+2|y_0|}\right\}$. If $\boldsymbol{x} = (x, y)$ and $\boldsymbol{x}_0 - \boldsymbol{x}\| < \delta$ then $|x - x_0| < \delta$, $y - y_0 < \delta$, and $|x| \le 1 + |x_0|$. Hence

$$\begin{aligned}|xy - x_0y_0| &= |xy - xy_0 + xy_0 - x_0y_0| = |x(y - y_0) + (x - x_0)y_0| \\ &\le |x(y - y_0)| + |(x - x_0)y_0| < |x|\delta + \delta|y_0| \\ &\le (1 + |x_0|)\frac{\epsilon}{2 + 2|x_0|} + \frac{\epsilon}{2 + 2|y_0|}|y_0| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,. \qquad \square\end{aligned}$$

Lemma 3.7. *Let* $A \subseteq \mathbb{R}^n$, *let* $B \subseteq \mathbb{R}^m$, *let* $\boldsymbol{f} : A \to \mathbb{R}^k$, *let* $\boldsymbol{g} : B \to \mathbb{R}^\ell$. *If* $\boldsymbol{f}$ *is continuous at a point* $\boldsymbol{x}_0 \in A$ *and* $\boldsymbol{g}$ *is continuous at the point* $\boldsymbol{y}_0 \in B$ *then* $(\boldsymbol{f}, \boldsymbol{g}) : A \times B \to \mathbb{R}^{k+\ell}(\boldsymbol{x}, \boldsymbol{y}) \mapsto (\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y}))$ *is continuous at* $(\boldsymbol{x}_0, \boldsymbol{y}_0)$.

Proof. Given $\epsilon > 0$. Choose $\delta_1 > 0$ such that $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta_1 \Rightarrow \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0\| < \epsilon/2$ and choose $\delta_2 > 0$ such that $\|\boldsymbol{y} - \boldsymbol{y}_0\| < \delta_2 \Rightarrow \|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{y}_0\| < \epsilon/2$. Put $\delta = \min\{\delta_1, \delta_2\}$. We have $\|(\boldsymbol{x}, \boldsymbol{y})\| = \sqrt{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2}$ so if $\|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}_0, \boldsymbol{y}_0)\| < \delta$ then $\|\boldsymbol{x} - \boldsymbol{x}_0\|, \|\boldsymbol{y} - \boldsymbol{y}_0\| < \delta$ and $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta_1$ and $\|\boldsymbol{y} - \boldsymbol{y}_0\| < \delta_2$. Hence $\|f(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| < \epsilon/2$ and $\|g(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{g}_0)\| < \epsilon/2$. Finally

$$\begin{aligned}&\|(\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{g}(\boldsymbol{y})) - (\boldsymbol{f}(\boldsymbol{x}_0), \boldsymbol{g}(\boldsymbol{y}_0))\| = \|(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0), \boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{y}_0))\| \\ &\qquad = \sqrt{\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\|^2 + \|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{y}_0)\|^2} < \sqrt{\frac{\epsilon^2}{4} + \frac{\epsilon^2}{4}} = \sqrt{\frac{\epsilon^2}{2}} < \epsilon\,. \qquad \square\end{aligned}$$

Composition of continuous functions are continuous:

Theorem 3.8. *Let* $A \subseteq \mathbb{R}^n$, *let* $B \subseteq \mathbb{R}^m$, *let* $\boldsymbol{f} : A \to \mathbb{R}^m$, *let* $\boldsymbol{g} : B \to \mathbb{R}^k$, *and assume that* $f(A) \subseteq B$. *If* $\boldsymbol{f}$ *is continuous at a point* $\boldsymbol{x}_0 \in A$ *and* $\boldsymbol{g}$ *is continuous at the point* $\boldsymbol{y}_0 = \boldsymbol{f}(x_0)$ *then* $\boldsymbol{g} \circ \boldsymbol{f}$ *is continuous at* $\boldsymbol{x}_0$.

Proof. Given $\epsilon > 0$. As $\boldsymbol{g}$ is continuous at $\boldsymbol{y}_0$ we can choose $\delta_1 > 0$ such that $\|\boldsymbol{y} - \boldsymbol{y}_0\| < \delta_1 \Rightarrow \|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{y}_0)\| < \epsilon$. As $\boldsymbol{f}$ is continuous at $\boldsymbol{x}_0$ we can choose $\delta > 0$ such that $\|\boldsymbol{x} - \boldsymbol{x}_0\| < \delta \Rightarrow \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| < \delta_1$. But then $\|\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0)) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) = \|\boldsymbol{g}(\boldsymbol{y}_0) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}))\| < \epsilon$ $\square$

Continuity is preserved by the usual arithmetic operations:

Theorem 3.9. *If* $\boldsymbol{x}_0 \in A$ *and* $\boldsymbol{f} : A \to \mathbb{R}^m$ *and* $\boldsymbol{g} : A \to \mathbb{R}^m$ *are continuous in* $\boldsymbol{x}_0$ *then*

1. $\boldsymbol{f}+\boldsymbol{g} : A \to \mathbb{R} : \boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})$ *is continuous in* $\boldsymbol{x}_0$.

If $\boldsymbol{x}_0 \in A$ *and* $f : A \to \mathbb{R}$ *and* $g : A \to \mathbb{R}$ *are continuous in* $\boldsymbol{x}_0$ *then*

2. $fg : A \to \mathbb{R} : \boldsymbol{x} \mapsto f(\boldsymbol{x})g(\boldsymbol{x})$ *is continuous in* x_0.

If $\boldsymbol{x}_0 \in A \subseteq \mathbb{R}^n$, $f : A \to \mathbb{R}$, $f(x) \neq 0$ *for all* $\boldsymbol{x} \in A$, *and* f *is continuous in* $\boldsymbol{x}_0$ *then*

3. $\frac{1}{f} : A \to \mathbb{R} : \boldsymbol{x} \mapsto \frac{1}{f(\boldsymbol{x})}$ *is continuous in* $\boldsymbol{x}_0$.

Proof of 1. We can write the vector function $\boldsymbol{f}+\boldsymbol{g}$ as the composition:

$$\boldsymbol{x} \stackrel{\text{diag}}{\longmapsto} (\boldsymbol{x},\boldsymbol{x}) \stackrel{(\boldsymbol{f},\boldsymbol{g})}{\longmapsto} (\boldsymbol{f}(\boldsymbol{x}),\boldsymbol{g}(\boldsymbol{x})) \stackrel{\text{add}}{\longmapsto} \boldsymbol{f}(\boldsymbol{x})+\boldsymbol{g}(\boldsymbol{x})\,,$$

By Lemma 3.4, Lemma 3.7, and Lemma 3.5 these three vector functions are continuous. By Theorem 3.8 the composition $\boldsymbol{f}+\boldsymbol{g} = \text{add} \circ (\boldsymbol{f},\boldsymbol{g}) \circ \text{diag}$ is continuous too. ☐

Proof of 2. We can write the function fg as the composition:

$$\boldsymbol{x} \stackrel{\text{diag}}{\longmapsto} (\boldsymbol{x},\boldsymbol{x}) \stackrel{(f,g)}{\longmapsto} (f(\boldsymbol{x}),g(\boldsymbol{x})) \stackrel{\text{mult}}{\longmapsto} f(\boldsymbol{x})g(\boldsymbol{x})\,,$$

By Lemma 3.4, Lemma 3.7, and Lemma 3.6 these three vector functions are continuous. By Theorem 3.8 the composition $fg = \text{mult} \circ (f,g) \circ \text{diag}$ is continuous too. ☐

Proof of 3. We can write the function $1/f$ as the composition:

$$\boldsymbol{x} \stackrel{f}{\longmapsto} f(\boldsymbol{x}) \stackrel{\text{inv}}{\longmapsto} \frac{1}{f(\boldsymbol{x})}\,,$$

where the last map is continuous by Lemma 2.4. ☐

We can check continuity of a vector function by looking at each coordinate separately:

Theorem 3.10. *A vector function* $\boldsymbol{f} = (f_1,\ldots,f_m) : A \to \mathbb{R}^m$ *is continuous at* $\boldsymbol{x}_0$ *if and only if all the functions* $f_k : A \to \mathbb{R}$, $k = 1,\ldots,m$ *are continuous at* $\boldsymbol{x}_0$.

Proof. We can write the coordinate function f_k as the composition $f_k = P_k \circ \boldsymbol{f}$ so if $\boldsymbol{f}$ is continuous at $\boldsymbol{x}_0$ then so is f_k. Conversely, assume all the coordinate functions f_k are continuous at $\boldsymbol{x}_0$ and we are given an $\epsilon > 0$. For each functions f_k we can find a $\delta_k > 0$ such that $\|\boldsymbol{x}-\boldsymbol{x}_0\| < \delta_k \Rightarrow |f_k(\boldsymbol{x}) - f_k(\boldsymbol{x}_0)| < \epsilon/m$. We now put $\delta = \min\{\delta_1,\ldots,\delta_m\}$. If $\|\boldsymbol{x}-\boldsymbol{x}_0\| < \delta$ then $|f_k(\boldsymbol{x}) - f_k(\boldsymbol{x}_0)| < \epsilon/m$ for all $k = 1,\ldots,m$ and hence

$$\begin{aligned}\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x}_0)\| &= \sqrt{|f_1(\boldsymbol{x}) - f_1(\boldsymbol{x}_0)|^2 + \cdots + |f_m(\boldsymbol{x}) - f_m(\boldsymbol{x}_0)|^2} \\ &< \sqrt{m\left(\frac{\epsilon}{m}\right)^2} = \frac{\epsilon}{\sqrt{m}} < \epsilon\,. \end{aligned}$$ ☐

3.3 Differentiability

Before we can define differentiability of vector functions $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ we need the concepts of *open sets*, (an analogue of open intervals).

Definition 3.11. The open ball in $\mathbb{R}^n$ with centre $\boldsymbol{x}$ and radius r is the subset $B(\boldsymbol{x}, r)$ consisting of points with a distance to $\boldsymbol{x}$ that is strictly smaller than r, i.e.,

$$B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y} - \boldsymbol{x}\| < r\}, \tag{3.9}$$

see Figure 3.2.

Definition 3.12. A subset $U \subseteq \mathbb{R}^n$ is called open if there for every point $\boldsymbol{x} \in U$ exists an open ball with centre $\boldsymbol{x}$ contained in U. That is

$$\forall \boldsymbol{x} \in U \, \exists r > 0 : B(\boldsymbol{x}, r) \subseteq U .$$

Loosely speaking: If we are in U then we can move a little bit in all direction and stay inside U, see Figure 3.2.

Example 3.5. An open ball is an open set. Indeed, consider $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$ put $\epsilon = r - \|\boldsymbol{x} - \boldsymbol{x}_0\|$. As $\|\boldsymbol{x} - \boldsymbol{x}_0\| < r$, we have $\epsilon > 0$. We want to show that $B(\boldsymbol{x}, \epsilon) \subset B(\boldsymbol{x}_0, r)$. Given $\boldsymbol{y} \in B(\boldsymbol{x}, \epsilon)$, i.e., $\|\boldsymbol{y} - \boldsymbol{x}\| < \epsilon$. Then

$$\begin{aligned}\|\boldsymbol{y} - \boldsymbol{x}_0\| = \|\boldsymbol{y} - \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_0\| \leq \|\boldsymbol{y} - \boldsymbol{x}_x\| + \|\boldsymbol{x} - \boldsymbol{x}_0\| \\ < \epsilon + \|\boldsymbol{x} - \boldsymbol{x}_0\| = r - \|\boldsymbol{x} - \boldsymbol{x}_0\| + \|\boldsymbol{x} - \boldsymbol{x}_0\| = r ,\end{aligned}$$

and $\boldsymbol{y} \in B(\boldsymbol{x}_0, r)$. The closed ball $\{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y} - \boldsymbol{x}\| \leq r\}$ is not open, see Figure 3.2, right. If $\boldsymbol{x}$ is on the boundary then, no matter how small ϵ is, a ball with centre $\boldsymbol{x}$

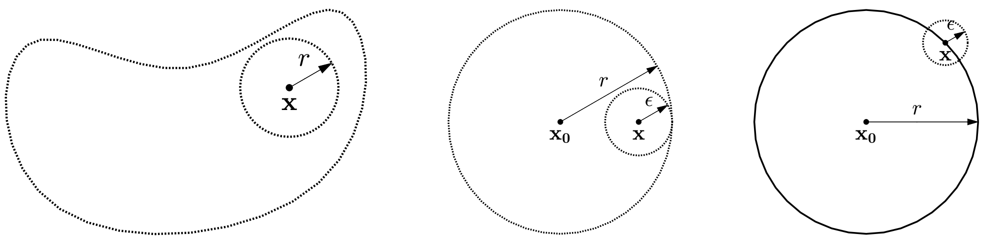

Figure 3.2: To the left an open set, in the middle an open ball, and to the right a closed ball.

and radius ϵ contains outside points.

Now we can define differentiability of vector functions:

Definition 3.13. Let $U \subseteq \mathbb{R}^n$ be an open set. A vector function $\boldsymbol{f} : U \to \mathbb{R}^m$ is called *differentiable* at a point $\boldsymbol{x} \in U$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\frac{\|\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - L(\boldsymbol{h})\|}{\|\boldsymbol{h}\|} \to 0 , \quad \text{for } \boldsymbol{h} \to \boldsymbol{0} . \tag{3.10}$$

Compare with (2.8). If $B(\boldsymbol{x}, r) \subseteq U$ we can write this as

$$\forall \epsilon > 0 \, \exists \delta \in]0, r] \, \forall \boldsymbol{h} \neq \boldsymbol{0} : \|\boldsymbol{h}\| < \delta \implies \frac{\|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - L(\boldsymbol{h})\|}{\|\boldsymbol{h}\|} < \epsilon \,.$$

The map L is called the *differential* of $\boldsymbol{f}$ at $\boldsymbol{x}$ and is denoted $\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}$. From Lemma 10.28 in [1] we know that L is of the form $L(\boldsymbol{h}) = \boldsymbol{J}\boldsymbol{h}$ where $\boldsymbol{J} \in \mathbb{R}^{m\times n}$ is a unique matrix called the *Jacobian matrix*.

If we put

$$\boldsymbol{\epsilon}(\boldsymbol{h}) = \frac{\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}_0) - L(\boldsymbol{h})}{\|\boldsymbol{h}\|} \,, \tag{3.11}$$

then (3.10) says that $\boldsymbol{\epsilon}(\boldsymbol{h}) \to \boldsymbol{0}$ for $\boldsymbol{h} \to \boldsymbol{0}$ and we have the following generalisation of (2.10):

$$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{f}(\boldsymbol{x}) + L(\boldsymbol{h}) + \boldsymbol{\epsilon}(\boldsymbol{h})\|\boldsymbol{h}\| \,. \tag{3.12}$$

In terms of the Jacobian matrix (3.12) becomes

$$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{J}\boldsymbol{h} + \boldsymbol{\epsilon}(\boldsymbol{h})\|\boldsymbol{h}\| \,, \tag{3.13}$$

and we see that for small $\boldsymbol{h}$ we can approximate $\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})$ by a first degree polynomial (in n variables) and the error goes to zero faster than $\|\boldsymbol{h}\|$.

Any linear map $L : \mathbb{R} \to \mathbb{R}^m$ can be written as $L(t) = tL(1)$. That leads us to the following definition

Definition 3.14. Let $I \subseteq \mathbb{R}$ be an open interval and let $\boldsymbol{f} : I \to \mathbb{R}^m$ be a vector function. If $\boldsymbol{f}$ is differentiable in $x \in I$ then we define $\boldsymbol{f}'(x) \in \mathbb{R}^m$ by $\boldsymbol{f}'(x) = \mathrm{d}\boldsymbol{f}_x(1)$. Then the differential $\mathrm{d}\boldsymbol{f}_x : \mathbb{R} \to \mathbb{R}^m$ can be written as $t \mapsto t\boldsymbol{f}'(x)$. If we consider the elements of $\mathbb{R}^n$ as column vectors then the Jacobian matrix is $\boldsymbol{J}(x) = \boldsymbol{f}'(x)$.

Lemma 3.15. *If $\boldsymbol{f} : I \to \mathbb{R}^m$ is differentiable in $x \in I$ then*

$$\boldsymbol{f}'(x) = \lim_{h\to 0} \frac{\boldsymbol{f}(x+h) - \boldsymbol{f}(x)}{h} \,. \tag{3.14}$$

Proof. This is a straight forward calculation. If $h \neq 0$ then

$$\begin{aligned}\left|\frac{\boldsymbol{f}(x+h) - \boldsymbol{f}(x)}{h} - \boldsymbol{f}'(x)\right| &= \left|\frac{\boldsymbol{f}(x+h) - \boldsymbol{f}(x) - \boldsymbol{f}'(x)h}{h}\right| \\ &= \frac{|\boldsymbol{f}(x+h) - \boldsymbol{f}(x) - \boldsymbol{f}'(x)h|}{|h|} \to 0 \quad \text{for } h \to 0 \,. \quad \square\end{aligned}$$

A linear map is differentiable:

Lemma 3.16. *If $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear then it is differentiable at all points $\boldsymbol{x} \in \mathbb{R}^n$ and the differential is $\mathrm{d}L_{\boldsymbol{x}} = L$.*

Proof. If $\boldsymbol{h} \neq \boldsymbol{0}$ then $\frac{\|L(\boldsymbol{x}+\boldsymbol{h}) - L(\boldsymbol{x}) - L(\boldsymbol{h})\|}{\|\boldsymbol{h}\|} = \boldsymbol{0}$. $\square$

Example 3.6. Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{b} \in \mathbb{R}^{1\times n}$, and $c \in \mathbb{R}$. A quadratic function $f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{x} + c$ is differentiable at all points $\boldsymbol{x} \in \mathbb{R}^n$ and the differential is given by $\mathrm{d}f_{\boldsymbol{x}}(\boldsymbol{h}) = \boldsymbol{x}^T(\boldsymbol{A} + \boldsymbol{A}^T)\boldsymbol{h} + \boldsymbol{b}\boldsymbol{h}$. Indeed

$$\frac{|f(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - (\boldsymbol{x}^T(\boldsymbol{A}+\boldsymbol{A}^T)\boldsymbol{h} + \boldsymbol{b}\boldsymbol{h})|}{\|\boldsymbol{h}\|} =$$
$$\frac{|(\boldsymbol{x}+\boldsymbol{h})^T\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{h}) + \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{h}) + c - \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\boldsymbol{x} - c - \boldsymbol{x}^T(\boldsymbol{A}+\boldsymbol{A}^T)\boldsymbol{h} + \boldsymbol{b}\boldsymbol{h}|}{\|\boldsymbol{h}\|}$$
$$= \frac{|\boldsymbol{h}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{h}^T\boldsymbol{A}\boldsymbol{h} - \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{h}|}{\|\boldsymbol{h}\|} = \frac{|\boldsymbol{h}^T\boldsymbol{A}\boldsymbol{h}|}{\|\boldsymbol{h}\|} \leq \frac{\|\boldsymbol{h}^T\|\|\boldsymbol{A}\boldsymbol{h}\|}{\|\boldsymbol{h}\|} = \|\boldsymbol{A}\boldsymbol{h}\| \to 0\,,$$

for $\boldsymbol{h} \to \boldsymbol{0}$. Where we at the end used that $\boldsymbol{h}^T\boldsymbol{A}\boldsymbol{x}$ is a number and hence $\boldsymbol{h}^T\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{h}^T\boldsymbol{A}\boldsymbol{x})^T = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{h}$, that a linear map is continuous and also Cauchy-Schwartz's inequality ($|\boldsymbol{x}^T\boldsymbol{y}| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|$). If $\boldsymbol{A}$ is symmetric the differential simplifies to $\mathrm{d}f_{\boldsymbol{x}}(\boldsymbol{h}) = 2\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{h} + \boldsymbol{b}\boldsymbol{h}$.

Just as in the case of functions of one variable, differentiability implies continuity:

Theorem 3.17. *Let $U \subseteq \mathbb{R}^n$ be an open set, let $\boldsymbol{f} : U \to \mathbb{R}^m$ be a vector function that is differentiable at a point $\boldsymbol{x} \in U$. Then $\boldsymbol{f}$ is continuous at $\boldsymbol{x}$.*

Proof. Let $\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}$ be the differential of $\boldsymbol{f}$ at $\boldsymbol{x}_0$ and choose $C \geq 0$ such that $\|\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h})\| < C\|\boldsymbol{h}\|$ for all $\boldsymbol{h} \in \mathbb{R}^n$. If $\epsilon > 0$ then we can choose $0 < \delta_0$ such that if $\|\boldsymbol{h}\| < \delta_0$ then $\boldsymbol{x} + \boldsymbol{h} \in U$ and $\frac{\|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{C}\boldsymbol{h}\|}{\|\boldsymbol{h}\|} < \frac{1}{2}\epsilon$. Put $\delta = \min\{\delta_0, \frac{\epsilon}{2C}\}$. If $\|\boldsymbol{h}\| < \delta$ then

$$\begin{aligned}\|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x})\| &= \|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h}) + \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h})\| \\ &\leq \|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h})\| + \|\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h})\| \\ &< \frac{\epsilon}{2} + C\|\boldsymbol{h}\| < \frac{\epsilon}{2} + C\frac{\epsilon}{2C} = \epsilon\,. \qquad \square\end{aligned}$$

Let $\boldsymbol{v} \in \mathbb{R}^n$ be a vector and consider the line $\boldsymbol{x} + t\boldsymbol{v}$. For small t we have $\boldsymbol{x} + t\boldsymbol{v} \in U$ and we can look at the restriction of $\boldsymbol{f}$ to these points.

Theorem 3.18. *Let $U \subseteq \mathbb{R}^n$ be an open set, let $\boldsymbol{f} : U \to \mathbb{R}^m$ be a vector function that is differentiable at a point $\boldsymbol{x} \in U$, and let $\boldsymbol{v} \in \mathbb{R}^n$. Then*

1. *There exists an $r > 0$ such that $\boldsymbol{x} + t\boldsymbol{v} \in U$ for all $t \in]-r, r[$.*

2. *The vector function $\boldsymbol{f}_{\boldsymbol{v}} :]-r, r[\to \mathbb{R}^m : t \mapsto \boldsymbol{f}(\boldsymbol{x} + t\boldsymbol{v})$ is differentiable at $t = 0$ and the derivative is $\boldsymbol{f}'_{\boldsymbol{v}}(0) = \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{v})$.*

The derivative of $\boldsymbol{f}_{\boldsymbol{v}}$ is called the directional derivative *in the direction $\boldsymbol{v}$ and is denoted $\partial_{\boldsymbol{v}}\boldsymbol{f}(\boldsymbol{x})$. That is $\partial_{\boldsymbol{v}}\boldsymbol{f}(\boldsymbol{x}) = \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{v})$.*

Proof. If $\boldsymbol{v} = \boldsymbol{0}$ then $\boldsymbol{x} + t\boldsymbol{v} = \boldsymbol{x} \in U$ for all t so r can be anything. Also $\boldsymbol{f}_{\boldsymbol{v}}$ is constant and $\boldsymbol{f}'_{\boldsymbol{0}}(0) = 0 = \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{0})$. So assume $\boldsymbol{v} \neq \boldsymbol{0}$. As U is open we can find an open ball $B(\boldsymbol{x}, r_0) \subseteq U$ if we put $r = \frac{r_0}{\|\boldsymbol{v}\|}$ then we have $|t| < r \Rightarrow \|\boldsymbol{x} + t\boldsymbol{v} - \boldsymbol{x}\| = \|t\boldsymbol{v}\| = |t|\|\boldsymbol{v}\| < r\|\boldsymbol{v}\| = r_0$. This proves 1. We now have

$$\frac{\|\boldsymbol{f_v}(t) - \boldsymbol{f_v}(0) - \mathrm{d}\boldsymbol{f_x}(\boldsymbol{v})t\|}{|t|} = \frac{\|\boldsymbol{f}(\boldsymbol{x} + t\boldsymbol{v}) - \boldsymbol{f}(\boldsymbol{x}) - \mathrm{d}\boldsymbol{f_x}(t\boldsymbol{v})\|}{\|t\boldsymbol{v}\||}\|\boldsymbol{v}\| \to 0\|\boldsymbol{v}\| = 0\,,$$

for $t \to 0$. This proves 2. □

In other words, the directional derivative is given by

$$\partial_{\boldsymbol{v}}\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f}(\boldsymbol{x}_0 + t\boldsymbol{v})\bigg|_{t=0}\,. \tag{3.15}$$

If we let $\boldsymbol{v}$ be one one of the basis vectors $\boldsymbol{e}_1 = (1, 0, \dots, 0)$, …, $\boldsymbol{e}_n = (0, \dots, 0, 1)$ then we obtain the *partial derivatives*:

Definition 3.19. The kth partial derivative at $\boldsymbol{x}$ is

$$\frac{\partial \boldsymbol{f}}{\partial x_k} = \partial_{\boldsymbol{e}_k}\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f}(\boldsymbol{x} + t\boldsymbol{e}_k)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{f}(x_1, \dots, x_k + t, \dots, x_n)\bigg|_{t=0}\,. \tag{3.16}$$

Definition 3.20. Let $U \subseteq \mathbb{R}^n$ be an open set. A vector function $\boldsymbol{f} : U \to \mathbb{R}^m$ is called *differentiable* if it is differentiable at all points $\boldsymbol{x} \in U$. The differential at $\boldsymbol{x}$ is denoted $\mathrm{d}\boldsymbol{f_x}$ and is a linear map $\mathbb{R}^n \to \mathbb{R}^m$. The differential $\mathrm{d}\boldsymbol{f}$ can be considered as a map $U \to \mathbb{R}^{m\times n} : \boldsymbol{x} \to \mathrm{d}\boldsymbol{f_x}$ and if it is continuous then f is called a C^1 vector function.

If I is an open interval and a function $f : I \to \mathbb{R}$ is differentiable with zero derivative $f' = 0$ then Corollary 2.30 tells us that f is constant. The same is true in higher dimensions.

Lemma 3.21. *If $B(\boldsymbol{x}_0, r) \subseteq \mathbb{R}^n$ is an open ball and a vector function $\boldsymbol{f} : B(\boldsymbol{x}_0, r) \to \mathbb{R}^m$ is differentiable with zero differential: $\mathrm{d}\boldsymbol{f_x}(\boldsymbol{v}) = \boldsymbol{0}$ for all $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$ then $\boldsymbol{f}$ is constant.*

Proof. If $\boldsymbol{x} \in B(\boldsymbol{x}_0, r)$ and $\boldsymbol{h} = \boldsymbol{x} - \boldsymbol{x}_0$ then the vector function $\boldsymbol{g} : t \mapsto \boldsymbol{f}(\boldsymbol{x}_0 + t\boldsymbol{h})$ has derivative $\boldsymbol{g}'(t) = \partial_{\boldsymbol{h}}\boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{h}) = \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}_0+\boldsymbol{h}}\boldsymbol{h} = \boldsymbol{0}$. Hence each coordinate function is constant, so $\boldsymbol{g}$ is constant, and $\boldsymbol{f}(x) = \boldsymbol{g}(1) = \boldsymbol{g}(0) = \boldsymbol{f}(\boldsymbol{x}_0)$. Hence $\boldsymbol{f}$ is constant too. □

We have seen that if a vector function is differentiable then the partial derivatives exists. The converse is not true, see Example 3.7, 3.8, and 3.9 below. But if the partial derivatives exists and are continuous we do have differentiability:

Theorem 3.22. *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\boldsymbol{f}$ be a vector function $U \to \mathbb{R}^m$. If the partial derivatives $\frac{\partial \boldsymbol{f}}{\partial x_k}$ exist at all points and are continuous then $\boldsymbol{f}$ is* *differentiable and the differential is given by* $\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}\boldsymbol{v} = \boldsymbol{J}\boldsymbol{v}$ *where* $\boldsymbol{J}$ *is the Jacobian matrix*

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial \boldsymbol{f}}{\partial x_1}(\boldsymbol{x}) & \dots & \frac{\partial \boldsymbol{f}}{\partial x_n}(\boldsymbol{x}) \end{pmatrix},$$

and we consider the partial derivatives $\frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x})$ *as column vectors. Furthermore, considered as a map* $\mathrm{d}\boldsymbol{f} : U \to \mathbb{R}^{m\times n} : \boldsymbol{x} \to \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}$ *the differential is continuous , i.e.,* $\boldsymbol{f}$ *is a* C^1 *vector function.*

Proof. Assume the partial derivatives exist and are continuous and let $\boldsymbol{x} \in U$. We want to show that $\boldsymbol{f}$ is differentiable in $\boldsymbol{x}$. First we choose $r > 0$ such that $B(\boldsymbol{x}, r) \subseteq U$. For $\boldsymbol{h} \in \mathbb{R}$ with $\|\boldsymbol{h}\| < r$ and for $k = 0, 1, \dots, n$ we put $\boldsymbol{x}_k = \boldsymbol{x} + \sum_{j=1}^{k} \boldsymbol{x} + h_j\boldsymbol{e}_j$, where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is the standard basis in $\mathbb{R}^n$ and $\boldsymbol{h} = (h_1, \dots, h_n)$. Then $\boldsymbol{x}_0 = \boldsymbol{x}$ and $\boldsymbol{x} + \boldsymbol{h} = \boldsymbol{x}_n$. Using the mean value theorem on the function $t \mapsto \boldsymbol{f}(\boldsymbol{x}_{k-1} + th_k\boldsymbol{e}_k)$ we have

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) &= \boldsymbol{f}(\boldsymbol{x}_n) - \boldsymbol{f}(\boldsymbol{x}_0)\\ &= \sum_{k=1}^{n}(\boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{f}(\boldsymbol{x}_{k-1})) = \sum_{k=1}^{n} \frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x}_{k-1} + \xi_k h_k \boldsymbol{e}_k)h_k\,,\end{aligned}$$

where $\xi_k \in]0, 1[$. Then

$$\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{J}\boldsymbol{h} = \sum_{k=1}^{n}\left(\frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x}_{k-1} + \xi_k h_k \boldsymbol{e}_k) - \frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x})\right)h_k\,,$$

and

$$\begin{aligned}\frac{\|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{J}\boldsymbol{h}\|}{\|\boldsymbol{h}\|} &= \sum_{k=1}^{n}\left\|\frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x}_{k-1} + \xi_k h_k \boldsymbol{e}_k) - \frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x})\right\|\frac{|h_k|}{\|\boldsymbol{h}\|}\\ &\le \sum_{k=1}^{n}\left\|\frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x}_{k-1} + \xi_k h_k \boldsymbol{e}_k) - \frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x})\right\|.\end{aligned}$$

We need to show that this goes to zero as $\boldsymbol{h}$ goes to zero. Given $\epsilon > 0$ we can choose $\delta_k \in]0, r[$ such that $\boldsymbol{h} < \delta_k \Rightarrow \left\|\frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x}+\boldsymbol{h}) - \frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x})\right\| < \epsilon/n$. We now put $\delta = \min\{\delta_1, \dots, \delta_n\}$. As $\|\xi_k h_k \boldsymbol{e}_k\| = |\xi h_k| \le h_k \le \|\boldsymbol{h}\|$ we see that if $\|\boldsymbol{h}\| < \delta$b then

$$\frac{\|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{J}\boldsymbol{h}\|}{\|\boldsymbol{h}\|} \le \sum_{k=1}^{n}\left\|\frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x}_{k-1} + \xi_k h_k \boldsymbol{e}_k) - \frac{\partial \boldsymbol{f}}{\partial x_k}(\boldsymbol{x})\right\| < \epsilon\,.$$

So $\boldsymbol{f}$ is differentiable at all $\boldsymbol{x} \in U$ and $\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}} = \boldsymbol{J}(\boldsymbol{x})$. The entries in $\boldsymbol{J}$ are the partial derivatives of the coordinate functions of $\boldsymbol{f}$ which are assumed to be continuous so $\mathrm{d}\boldsymbol{f}$ is continuous. □

The following examples show that the existence of all partial derivatives or all directional derivatives does not guarantee differentiability:

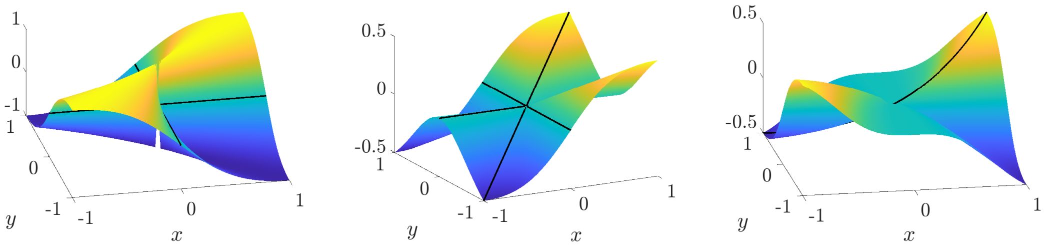

Figure 3.3: From left to right: Example 3.7, 3.8, and 3.9.

Example 3.7. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\,,\\ 0 & \text{for } (x,y)=(0,0)\,,\end{cases}$$

If $t\neq 0$ then

$$\frac{f(t,0)-f(0,0)}{t}=0\,,\quad\text{and}\quad\frac{f(0,t)-f(0,0)}{t}=0\,.$$

So both partial derivative exists at $(x,y)=(0,0)$ (and $\frac{\partial \boldsymbol{f}}{\partial x}(0,0)=\frac{\partial \boldsymbol{f}}{\partial y}(0,0)=0$). If $t\neq 0$ then $\boldsymbol{f}(t,t)=\frac{2tt}{t^2+t^2}=1$. So $\boldsymbol{f}$ is discontinuous at $(0,0)$ and consequently not differentiable, see Figure 3.3 left.

Example 3.8. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=\begin{cases}\dfrac{x\,y^2}{x^2+y^2} & \text{for } (x,y)\neq(0,0)\,,\\ 0 & \text{for } (x,y)=(0,0)\,.\end{cases}$$

Now f is continuous at $(0,0)$: If $(x,y)\neq(0,0)$ then we can write $(x,y)=(r\cos\theta,r\sin\theta)$ (polar coordinates) and then we have

$$|f(x,y)-f(0,0)|=\left|\frac{r\cos\theta r^2\sin^2\theta}{r^2\cos^2\theta+r^2\sin^2\theta}\right|=|r\cos\theta\sin^2\theta|\leq|r|\to 0\,,$$

for $(x,y)\to(0,0)$. We also have a directional derivative in all directions: If $(v,w)\neq(0,0)$ and $t\neq 0$ then

$$\frac{|f(tv,tw)-f(0,0)-vw^2t|}{|t|}=\frac{\left|\frac{tvt^2w^2}{t^2v^2+t^2w^2}-\frac{vw^2}{v^2+w^2}t\right|}{|t|}=0\,,$$

so $\partial_{(v,w)}f=\frac{vw^2}{v^2+w^2}$. But the map $(v,w)\mapsto\partial_{(v,w)}f=\frac{vw^2}{v^2+w^2}$ is not linear so f is not differentiable, see the middle picture in Figure 3.3.

Example 3.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^3\,y}{x^4+y^2} & \text{for } (x,y) \neq (0,0)\,, \\ 0 & \text{for } (x,y) = (0,0)\,. \end{cases}$$

Again f is continuous at $(0,0)$: If $(x,y) \neq (0,0)$ then we can write $(x,y) = (\sqrt{v}t, wt^2)$, where $v \geq 0$ and $v^2 + w^2 = 1$. Then $v, |w| \leq 1$ and if $|t| < 1$ we have $x^2 + y^2 = vt^2 + w^2t^4 \geq v^2t^2 + w^2t^2 = t^2$, i.e., $|t| \leq \|(x,y)\|$. Furthermore,

$$\begin{aligned} |f(x,y) - f(0,0)| = \left|\frac{x^3y}{x^4+y^2}\right| = \left|\frac{v\sqrt{v}t^3wt^2}{v^2t^4+w^2t^4}\right| = \left|\frac{v\sqrt{v}wt}{v^2+w^2}\right| \\ = v\sqrt{v}|w||t| \leq |t| = \|(x,y)\| \to 0 \quad \text{for } (x,y) \to (0,0)\,. \end{aligned}$$

We also have a directional derivative in all directions: If $(v,w) \neq (0,0)$ and $t \neq 0$ then

$$\frac{f(vt,wt) - f(0,0)}{t} = \frac{v^3t^3wt}{v^4t^5 + w^2t^3} = \frac{v^3wt}{v^4t^2+w^2} \to 0 \quad \text{for } t \to 0\,.$$

So $\partial_{(v,w)}f = 0$. We see that the map $(v,w) \mapsto \partial_{(v,w)}f = 0$ is linear, see Figure 3.3 right. If f were differentiable at $(0,0)$ then the differential $\mathrm{d}f_{(0,0)}$ would be the zero map. But, if $\boldsymbol{h} = (t, t^2)$ and $t \neq 0$ then

$$\frac{|f(\mathbf{0}+\boldsymbol{h}) - f(\mathbf{0}) - \mathbf{0}\boldsymbol{h}|}{\|\boldsymbol{h}\|} = \frac{|t^3t^2|}{(t^4+t^4)\sqrt{t^2+t^4}} = \frac{1}{2\sqrt{1+t^2}} \xrightarrow[t\to 0]{} \frac{1}{2}\,.$$

So f is not differentiable (the limit should be 0).

As the projection $P_k : \mathbb{R}^n \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto x_k$, the diagonal map $\operatorname{diag} : \mathbb{R}^n \to \mathbb{R}^{2n} : \boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{x})$, and addition $\operatorname{add} : \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} + \boldsymbol{y}$ all are linear maps Lemma 3.16 immediately give us

Corollary 3.23 (Projection on the kth coordinate)**.** *Let $k \in \{1, 2, \ldots, n\}$. The function $P_k : \mathbb{R}^n \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto x_k$ is differentiable at all points $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and the differential is P_k.*

Corollary 3.24 (Diagonal)**.** *The vector function* $\operatorname{diag} : \mathbb{R}^n \to \mathbb{R}^{2n} : \boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{x})$ *is differentiable at all points $\boldsymbol{x} \in \mathbb{R}^n$ and the differential is* diag *with matrix* $\begin{pmatrix} \boldsymbol{I}_n \\ \boldsymbol{I}_n \end{pmatrix}$, *where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix*

Corollary 3.25. *Addition* $\operatorname{add} : \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n : (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{x} + \boldsymbol{y}$ *is differentiable at all points $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n}$ and the differential is* add *with matrix* $\begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{I}_n \end{pmatrix}$.

We also have

Lemma 3.26. *Multiplication* $\operatorname{mult} : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto xy$ *is differentiable at all points $(x,y) \in \mathbb{R}^2$ and the differential* $\mathrm{d}\operatorname{mult}_{(x,y)}$ *has matrix* $\begin{pmatrix} y & x \end{pmatrix}$.

Proof. The partial derivatives are $\frac{\partial \operatorname{mult}}{\partial x} = y$ and $\frac{\partial \operatorname{mult}}{\partial y} = x$. We see they are continuous so by Theorem 3.22 mult is differentiable and the differential has the matrix $\left(\frac{\partial \operatorname{mult}}{\partial x} \; \frac{\partial \operatorname{mult}}{\partial y} \right) = \left(y \; x \right)$. □

Finally we have

Lemma 3.27. *Let $U \subseteq \mathbb{R}^n$, let $V \subseteq \mathbb{R}^m$ be open sets, let $\boldsymbol{f} : U \to \mathbb{R}^k$, let $\boldsymbol{g} : V \to \mathbb{R}^\ell$. If $\boldsymbol{f}$ is differentiable at a point $\boldsymbol{x} \in U$ and $\boldsymbol{g}$ is differentiable at the point $\boldsymbol{y} \in V$ then $(\boldsymbol{f}, \boldsymbol{g}) : U \times V \to \mathbb{R}^{k+\ell}(\boldsymbol{v}, \boldsymbol{w}) \mapsto (\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{g}(\boldsymbol{w}))$ is differentiable at $(\boldsymbol{x}, \boldsymbol{y})$ with differential $\mathrm{d}(\boldsymbol{f}, \boldsymbol{g})_{(\boldsymbol{x},\boldsymbol{y})} = (\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}, \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}})$. If $\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}$ has the matrix $\boldsymbol{A} \in \mathbb{R}^{n\times k}$ and $\mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}$ has the matrix $\boldsymbol{B} \in \mathbb{R}^{m\times \ell}$ then $\mathrm{d}(\boldsymbol{f}, \boldsymbol{g})_{(\boldsymbol{x},\boldsymbol{y})}$ has matrix $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{smallmatrix} \right)$.*

Proof. If we for a pair $(\boldsymbol{h}, \boldsymbol{k}) \in \mathbb{R}^n \times \mathbb{R}^m$ have that $(\boldsymbol{x} + \boldsymbol{h}, \boldsymbol{y} + \boldsymbol{k}) \in U \times V$ then

$$\begin{aligned} &\frac{\left\| \begin{pmatrix} \boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{h}) \\ \boldsymbol{g}(\boldsymbol{y}_0 + \boldsymbol{k}) \end{pmatrix} - \begin{pmatrix} \boldsymbol{f}(\boldsymbol{x}_0) \\ \boldsymbol{g}(\boldsymbol{y}_0)) \end{pmatrix} - \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix} \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{k} \end{pmatrix} \right\|}{\|(\boldsymbol{h}, \boldsymbol{k})\|} \\ &\qquad = \frac{\sqrt{\|\boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}_0) - \boldsymbol{A}\boldsymbol{h}\|^2 + \|\boldsymbol{g}(\boldsymbol{y}_0 + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}_0) - \boldsymbol{B}\boldsymbol{k}\|^2}}{\sqrt{\|\boldsymbol{h}\|^2 + \|\boldsymbol{k}\|^2}} \\ &\qquad \le \frac{\|\boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}_0) - \boldsymbol{A}\boldsymbol{h}\| + \|\boldsymbol{g}(\boldsymbol{y}_0 + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}_0) - \boldsymbol{B}\boldsymbol{k}\|}{\sqrt{\|\boldsymbol{h}\|^2 + \|\boldsymbol{k}\|^2}} \\ &\qquad = \frac{\|\boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}_0) - \boldsymbol{A}\boldsymbol{h}\|}{\sqrt{\|\boldsymbol{h}\|^2 + \|\boldsymbol{k}\|^2}} + \frac{\|\boldsymbol{g}(\boldsymbol{y}_0 + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}_0) - \boldsymbol{B}\boldsymbol{k}\|}{\sqrt{\|\boldsymbol{h}\|^2 + \|\boldsymbol{k}\|^2}} \to 0\,, \end{aligned}$$

for $(\boldsymbol{h}, \boldsymbol{k}) \to (\boldsymbol{0}, \boldsymbol{0})$. □

The composition of differentiable vector functions is differentiable:

Theorem 3.28. *If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open sets, $\boldsymbol{f} : U \to \mathbb{R}^m$ and $\boldsymbol{g} : V \to \mathbb{R}^k$ are vector functions such that $\boldsymbol{f}(U) \subset V$, $\boldsymbol{f}$ is differentiable in $\boldsymbol{x} \in U$ and $\boldsymbol{g}$ is differentiable in $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}_0)$. Then $\boldsymbol{g} \circ \boldsymbol{f} : U \to \mathbb{R}^k$ is differentiable in $\boldsymbol{x}$ with differential $\mathrm{d}(\boldsymbol{g} \circ \boldsymbol{f})_{\boldsymbol{x}} = \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}} \circ \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}$.*

Proof. For an $\boldsymbol{h} \in \mathbb{R}^n$ with $\boldsymbol{x}_0 + \boldsymbol{h} \in U$ we put

$$\boldsymbol{\epsilon}_1(\boldsymbol{h}) = \frac{\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h})}{\|\boldsymbol{h}\|}\,.$$

Then $\|\boldsymbol{\epsilon}_1(\boldsymbol{h})\| \to 0$ for $\boldsymbol{h} \to \boldsymbol{0}$ and $\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) = \boldsymbol{f}(\boldsymbol{x}) + \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h}) + \|\boldsymbol{h}\|\boldsymbol{\epsilon}_1(\boldsymbol{h})$. Similar, for a $\boldsymbol{k} \in \mathbb{R}^m$ with $\boldsymbol{y} + \boldsymbol{k} \in V$ we put

$$\boldsymbol{\epsilon}_2(\boldsymbol{k}) = \frac{\boldsymbol{g}(\boldsymbol{y}_0 + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}_0) - \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{k})}{\|\boldsymbol{k}\|}\,,$$

and then $\|\boldsymbol{\epsilon}_2(\boldsymbol{k})\| \to 0$ for $\boldsymbol{k} \to \mathbf{0}$ and $\boldsymbol{g}(\boldsymbol{y}+\boldsymbol{k}) = \boldsymbol{g}(\boldsymbol{x}) + \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{k}) + \|\boldsymbol{k}\|\boldsymbol{\epsilon}_2(\boldsymbol{k})$. If we let $\boldsymbol{k} = \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h}) + \|\boldsymbol{h}\|\boldsymbol{\epsilon}_1(\boldsymbol{h})$ then $\boldsymbol{k} \to \mathbf{0}$ for $\boldsymbol{h} \to \mathbf{0}$ and we have

$$\begin{aligned}
&\frac{\|\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h})) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) - \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}(\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h}))\|}{\|\boldsymbol{h}\|} \\
&\qquad = \left\| \frac{\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}) + \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h}) + \|\boldsymbol{h}\|\boldsymbol{\epsilon}_1(\boldsymbol{h})) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) - \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}(\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{h}))}{\|\boldsymbol{h}\|} \right\| \\
&\qquad = \left\| \frac{\|\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x})) - \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{k} - \|\boldsymbol{h}\|\boldsymbol{\epsilon}_1(\boldsymbol{h}))}{\|\boldsymbol{h}\|} \right\| \\
&\qquad = \left\| \frac{\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0) + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{x}_0)) - \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{k})}{\|\boldsymbol{h}\|} + \frac{\mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}(\|\boldsymbol{h}\|\boldsymbol{\epsilon}_1(\boldsymbol{h}))}{\|\boldsymbol{h}\|} \right\| \\
&\qquad\qquad = \|\boldsymbol{\epsilon}_2(\boldsymbol{k}) + \mathrm{d}\boldsymbol{g}_{\boldsymbol{y}}(\boldsymbol{\epsilon}_1(\boldsymbol{h}))\| \to 0 \text{ for } \boldsymbol{h} \to \mathbf{0}\,. \quad \square
\end{aligned}$$

Example 3.10. Let $U \subseteq \mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be differentiable. If $(x, y) \in U$ and $(x + tv, y + tw) \in U$ for all $t \in [0, 1]$ then we have

$$\frac{\mathrm{d}f}{\mathrm{d}t}(x+tv, y+tw) = v\frac{\partial f}{\partial x}(x+tv, y+tw) + w\frac{\partial f}{\partial y}(x+tv, y+tw)\,,$$

or short $\frac{\mathrm{d}f}{\mathrm{d}t} = v\frac{\partial f}{\partial x} + w\frac{\partial f}{\partial y}$.

Example 3.11. Let $U \subseteq \mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be differentiable. If $\boldsymbol{x} \in U$ and $\boldsymbol{x} + t\boldsymbol{h} \in U$ for all $t \in [0, 1]$ then we have

$$\frac{\mathrm{d}f}{\mathrm{d}t}(\boldsymbol{x}+t\boldsymbol{h}) = \frac{\partial f}{\partial x_1}(\boldsymbol{x}+t\boldsymbol{h})h_1 + \cdots + \frac{\partial f}{\partial x_n}(\boldsymbol{x}+t\boldsymbol{h})h_n\,,$$

where $\boldsymbol{h} = (h_1, \ldots, h_n)$. Short $\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} h_i$.

Just like continuity, differentiability is preserved by the usual arithmetic operations:

Theorem 3.29. *If $\boldsymbol{x} \in U$ and $\boldsymbol{f} : U \to \mathbb{R}^m$ and $\boldsymbol{g} : U \to \mathbb{R}^m$ are differentiable in $\boldsymbol{x}$ then*

1. *$\boldsymbol{f} + \boldsymbol{g} : A \to \mathbb{R} : \boldsymbol{v} \mapsto \boldsymbol{f}(\boldsymbol{v}) + \boldsymbol{g}(\boldsymbol{v})$ is differentiable in $\boldsymbol{x}$ with differential $\mathrm{d}(\boldsymbol{f}+\boldsymbol{g})_{\boldsymbol{x}} = \mathrm{d}(\boldsymbol{f}+\boldsymbol{g})_{\boldsymbol{x}} = \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}} + \mathrm{d}\boldsymbol{g}_{\boldsymbol{x}}$.*

If $\boldsymbol{x} \in U$ and $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are differentiable in $\boldsymbol{x}$ then

2. *$fg : U \to \mathbb{R} : \boldsymbol{v} \mapsto f(\boldsymbol{v})g(\boldsymbol{v})$ is differentiable in $\boldsymbol{x}$ with differential $\mathrm{d}(fg)_{\boldsymbol{x}} = g(\boldsymbol{x})\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}} + f(\boldsymbol{x})\mathrm{d}\boldsymbol{g}_{\boldsymbol{x}}$.*

If $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}$, $f(\boldsymbol{v}) \neq 0$ for all $\boldsymbol{v} \in A$, and f is differentiable in $\boldsymbol{x} \in U$ then

3. *$\frac{1}{f} : U \to \mathbb{R} : \boldsymbol{v} \mapsto \frac{1}{f(\boldsymbol{v})}$ is differentiable in $\boldsymbol{x}$ with differential $-f(\boldsymbol{x})^{-2}\mathrm{d}f_{\boldsymbol{x}}$.*

Proof of 1. We can write the vector function $\boldsymbol{f}+\boldsymbol{g}$ as the composition:

$$\boldsymbol{v} \overset{\text{diag}}{\longmapsto} (\boldsymbol{v},\boldsymbol{v}) \overset{(\boldsymbol{f},\boldsymbol{g})}{\longmapsto} (\boldsymbol{f}(\boldsymbol{v}),\boldsymbol{g}(\boldsymbol{v})) \overset{\text{add}}{\longmapsto} \boldsymbol{f}(\boldsymbol{v})+\boldsymbol{g}(\boldsymbol{v}),$$

By Corollary 3.24, Lemma 3.27, and corollary 3.25 these three vector functions are continuous. By Theorem 3.28 the composition $\boldsymbol{f}+\boldsymbol{g} = \operatorname{add}\circ(\boldsymbol{f},\boldsymbol{g})\circ\operatorname{diag}$ is differentiable too and the differential is

$$\mathrm{d}(\boldsymbol{f}+\boldsymbol{g})_{\boldsymbol{x}_0} = \operatorname{add}\circ(\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}},\mathrm{d}\boldsymbol{g}_{\boldsymbol{x}})\circ\operatorname{diag} = \mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}+\mathrm{d}\boldsymbol{g}_{\boldsymbol{x}}\,. \qquad \square$$

Proof of 2. We can write the function fg as the composition:

$$\boldsymbol{v} \overset{\text{diag}}{\longmapsto} (\boldsymbol{v},\boldsymbol{v}) \overset{(f,g)}{\longmapsto} (f(\boldsymbol{v}),g(\boldsymbol{v})) \overset{\text{mult}}{\longmapsto} f(\boldsymbol{v})g(\boldsymbol{v}),$$

By Corollary 3.24, Lemma 3.27, and Lemma 3.26 these three vector functions are differentiable. By Theorem 3.8 the composition $fg = \operatorname{mult}\circ(f,g)\circ\operatorname{diag}$ is differentiable too and the differential is

$$\mathrm{d}(f+g)_{\boldsymbol{x}_0} = \mathrm{d}\operatorname{mult}_{(f(\boldsymbol{x}),g(\boldsymbol{x}))}\circ(\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}},\mathrm{d}\boldsymbol{g}_{\boldsymbol{x}})\circ\operatorname{diag} = g(\boldsymbol{x})\mathrm{d}\boldsymbol{f}_{\boldsymbol{x}}+f(\boldsymbol{x})\mathrm{d}\boldsymbol{g}_{\boldsymbol{x}}\,. \qquad \square$$

Proof of 3. We can write the function $1/f$ as the composition:

$$\boldsymbol{v} \overset{f}{\longmapsto} f(\boldsymbol{v}) \overset{\text{inv}}{\longmapsto} \frac{1}{f(\boldsymbol{v})},$$

where the last map is differentiable by Lemma 2.18. By Theorem 3.8 the composition is differentiable too and the differential is

$$\mathrm{d}\left(\frac{1}{f}\right)_{\boldsymbol{x}} = \mathrm{d}\operatorname{inv}_{f(\boldsymbol{x})}\circ\mathrm{d}f_{\boldsymbol{x}} = \frac{-\mathrm{d}f_{\boldsymbol{x}}}{f(\boldsymbol{x})^2}\,. \qquad \square$$

Just as in the case of functions of one variable we can define higher order differentiability recursively:

Definition 3.30. Let $U \subseteq \mathbb{R}^n$ be an open set. A vector function $\boldsymbol{f}: U \to \mathbb{R}^m$ is k *times differentiable* if $\boldsymbol{f}$ is differentiable and the differential considered as a function $\mathrm{d}\boldsymbol{f}: U \to \mathbb{R}^{m\times n}$ is $k-1$ times differentiable. If the kth derivative is continuous then $\boldsymbol{f}$ is called a C^k *function* If $\boldsymbol{f}$ is a C^k function for all $k \in \mathbb{N}$ then $\boldsymbol{f}$ is called a C^∞ *function*

In terms of the partial derivatives we have

Theorem 3.31. *Let $U \subseteq \mathbb{R}^n$ be an open set. A vector function $\boldsymbol{f}: U \to \mathbb{R}^m$ is a C^k vector function if and only if all partial derivatives up to order k exists and are continuous.*

Proof. If $\boldsymbol{f}$ is a C^k vector function then is clear that the partial derivatives of order k exists and are continuous.

We need to show that if all partial derivatives of order k exists and are continuous then we have a C^k function. We prove it by induction on the order k. The case $k = 1$ is Theorem 3.22. So now assume that the theorem is true for a $k \in \mathbb{N}$ and that $\boldsymbol{f}$ is a function where all partial derivatives of order $k+1$ exist and are continuous. As $k \geq 1$ Theorem 3.22 shows that $\boldsymbol{f}$ is a C^1 function, so $\mathrm{d}\boldsymbol{f} : U \to \mathbb{R}^{m\times n}$ exists and are continuous. As all partial derivatives of $\boldsymbol{f}$ up to order $k+1$ exist and are continuous, all partial derivatives of $\mathrm{d}\boldsymbol{f}$ up to order k exist and are continuous. So by the induction hypothesis $\mathrm{d}\boldsymbol{f}$ is a C^k vector function, but that mean by definition that $\boldsymbol{f}$ is a C^{k+1} vector function. □

If $\boldsymbol{f}$ is a C^k vector function then the order of differentiation does not matter. We first consider a function of two variables.

Lemma 3.32. *If $U \subseteq \mathbb{R}^2$ is a open set and $\boldsymbol{f} : U \to \mathbb{R}$ is a C^2 function, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.*

Proof. Consider a rectangle $[a, b] \times [c, d] \subseteq U$ and the function

$$g_1(x) = f(x, d) - f(x, c)\,, \quad x \in [a, b]\,.$$

We have

$$g_1(b) - g_1(a) = f(b, d) - f(b, c) - f(a, d) + f(a, c)\,.$$

The function g_1 is C^2 with derivative $g_1'(x) = \frac{\partial f}{\partial x}(x, d) - \frac{\partial f}{\partial x}(x, c)$. The mean value theorem gives us a $\xi_1 \in [a, b]$ such that $g_1(b) - g_1(a) = g_1'(\xi_1)(b - a)$ and hence

$$\left(\frac{\partial f}{\partial x}(\xi_1, d) - \frac{\partial f}{\partial x}(\xi_1, c)\right)(b - a) = f(b, d) - f(b, c) - f(a, d) + f(a, c)\,.$$

Next we consider the function

$$h_1(y) = \frac{\partial f}{\partial x}(\xi_1, y)\,, \quad y \in [c, d]\,.$$

The function h_1 is C^1 with derivative $\frac{\partial^2 f}{\partial y \partial x}(\xi_1, y)$ and the mean value theorem gives us $\eta_1 \in [c, d]$ such that $\frac{\partial f}{\partial x}(\xi_1, d) - \frac{\partial f}{\partial x}(\xi_1, c) = \frac{\partial^2 f}{\partial y \partial x}(\xi_1, \eta_1)(d - c)$. We now have

$$\frac{\partial^2 f}{\partial y \partial x}(\xi_1, \eta_1)(d - c)(b - a) = f(b, d) - f(b, c) - f(a, d) + f(a, c)\,. \qquad (*)$$

If we instead start with the function

$$g_2(y) = f(b, y) - f(a, y)\,, \quad y \in [c, d]\,.$$

We have

$$g_2(d) - g_2(c) = f(b, d) - f(a, d) - f(b, c) + f(a, c)\,,$$

the same expression as for $g_1(b) - g_1(a)$. The function g_2 is C^2 with derivative $g_2'(y) = \frac{\partial f}{\partial y}(b, y) - \frac{\partial f}{\partial y}(a, y)$. The mean value theorem gives us a $\eta_2 \in [c, d]$ such that $g_2(d) - g_1(c) = g_2'(\eta_2)(d - c)$ and hence

$$\left(\frac{\partial f}{\partial y}(b, \eta_2) - \frac{\partial f}{\partial y}(a, \eta_2)\right)(d - c) = f(b, d) - f(b, c) - f(a, d) + f(a, c)\,.$$

Continuing as before we look at the function

$$h_2(x) = \frac{\partial f}{\partial y}(x, \eta_2)\,, \quad x \in [a, b]\,.$$

The function h_2 is C^1 with derivative $\frac{\partial^2 f}{\partial x \partial y}(x, \eta_2)$ and the mean value theorem gives us $\xi_2 \in [a, b]$ such that $\frac{\partial f}{\partial y}(b, \eta_2) - \frac{\partial f}{\partial y}(a, \eta_2) = \frac{\partial^2 f}{\partial x \partial y}(\xi_2, \eta_2)(b - a)$. We now have

$$\frac{\partial^2 f}{\partial x \partial y}(\xi_2, \eta_2)(b - a)(c - d) = f(b, d) - f(b, c) - f(a, d) + f(a, c)\,.$$

Comparing with (∗) we see that

$$\frac{\partial^2 f}{\partial x \partial y}(\xi_2, \eta_2) = \frac{\partial^2 f}{\partial y \partial x}(\xi_1, \eta_1)\,, \qquad (**)$$

where $(\xi_k, \eta_k) \in [a, b] \times [c, d]$. Given $(x, y) \in U$ we can find $r > 0$ such that $[x - r, x + r] \times [y - r, y + r] \subseteq U$. For any $\epsilon \in]0, r]$ we have just seen that there exists $(\xi_k, \eta_k) \in [x - \epsilon, x + \epsilon] \times [y - \epsilon, y + \epsilon]$ for $k = 1, 2$ such that (∗∗) holds. If $\epsilon \to 0$ then $(\xi_k, \eta_k) \to (x, y)$ for $k = 1, 2$ and as f is C^2 the functions $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous. So in the limit we have

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)\,. \qquad \square$$

Theorem 3.33. *If $U \subseteq \mathbb{R}^n$ is a open set and $\boldsymbol{f} : U \to \mathbb{R}^m$ is a C^2 vector function, then $\frac{\partial^2 \boldsymbol{f}}{\partial x_i \partial x_j} = \frac{\partial^2 \boldsymbol{f}}{\partial x_j \partial x_i}$ for all $i, j = 1, \ldots, n$.*

Proof. If $i = j$ then there is nothing to show so we assume that $i < j$. If $\boldsymbol{f} = (f_1, \ldots, f_n)$ and given a point $\boldsymbol{x} = (x_1, \ldots, x_n) \in U$ we consider the functions

$$g_k(x, y) = f_k(x_1, \ldots, x_{i-1}, x, \ldots, x_{i+1}, \ldots, x_{j-1}, y, \ldots, x_{j+1}, \ldots, x_n)\,,$$

for $k = 1, \ldots, n$. By Lemma 3.32 we now have

$$\frac{\partial^2 f_k}{\partial x_i \partial x_j} = \frac{\partial^2 g_k}{\partial x \partial y} = \frac{\partial^2 g_k}{\partial y \partial x} = \frac{\partial^2 f_k}{\partial x_j \partial x_i}\,, \quad \text{for } k = 1, \ldots, n\,. \qquad \square$$

By induction on the order of differentiation the following follows

Theorem 3.34. *If $U \subseteq \mathbb{R}^n$ is a open set, $k \geq 2$, and $\boldsymbol{f} : U \to \mathbb{R}^m$ is a C^k vector function, then $\frac{\partial^k \boldsymbol{f}}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial^k \boldsymbol{f}}{\partial x_{i_{\sigma(k)}} \dots \partial x_{i_{\sigma(1)}}}$ for all $i_1, \dots, i_k \in \{1, \dots, n\}$, all permutations $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$.*

Proof. We use induction on the order k. The case $k = 2$ is Theorem 3.33. So now we assume the theorem holds for a $k \geq 2$ and consider a permutation $\sigma : \{1, \dots, k+1\} \to \{1, \dots, k+1\}$. If $\sigma(1) \neq k+1$ then $k+1 = \sigma(\ell)$ for an $\ell \in \{2, \dots, k+1\}$. Let $\tau : \{2, \dots, k+1\} \to \{2, \dots, k+1\}$ be the permutation that exchange ℓ and k, i.e., $\tau(\ell) = k+1$, $\tau(k+1) = \ell$, and $\tau(j) = j$ otherwise. Then using the induction hypothesis on the vector function $\frac{\partial f}{\partial x_{\sigma(1)}}$ we have

$$\frac{\partial^k}{\partial x_{i_{\sigma(k+1)}} \dots \partial x_{i_{\sigma(2)}}} \frac{\partial f}{\partial x_{\sigma(1)}} = \frac{\partial^k}{\partial x_{i_{\sigma(\tau(k+1))}} \dots \partial x_{i_{\sigma(\tau(2))}}} \frac{\partial f}{\partial x_{\sigma(1)}},$$

and we let $\widehat{\sigma}\{1, \dots, k+1\} \to \{1, \dots, k+1\}$ be the permutation defined by $\widehat{\sigma}(1) = \sigma(1)$ and $\widehat{\sigma}(j) = \sigma(\tau(j))$ if $j \neq 1$. We have in particular that $\widehat{\sigma}(k+1) = \sigma(\tau(k+1)) = \sigma(\ell) = k+1$.

If $\sigma(1) = k+1$ then using Theorem 3.33 on the function $\boldsymbol{f}$ we have

$$\frac{\partial^k}{\partial x_{i_{\sigma(k+1)}} \dots \partial x_{i_{\sigma(3)}}} \frac{\partial^2 f}{\partial x_{\sigma(2)} \partial x_{\sigma(1)}} = \frac{\partial^k}{\partial x_{i_{\sigma(k+1)}} \dots \partial x_{i_{\sigma(3)}}} \frac{\partial^2 f}{\partial x_{\sigma(1)} \partial x_{\sigma(2)}}.$$

If we now define a permutation $\tau : \{1, 3, \dots, k+1\} \to \{1, 3, \dots, k+1\}$ by $\tau(1) = k+1$, $\tau(k+1) = 1$, and $\tau(j) = j$ otherwise. Then using the induction hypothesis on the vector function $\frac{\partial f}{\partial x_{\sigma(2)}}$ we have

$$\frac{\partial^k}{\partial x_{i_{\sigma(k+1)}} \dots \partial x_{i_{\sigma(3)}} \partial x_{\sigma(1)}} \frac{\partial f}{\partial x_{\sigma(2)}} = \frac{\partial^k}{\partial x_{\sigma(1)} \partial x_{i_{\sigma(k)}} \dots \partial x_{i_{\sigma(3)}} \partial x_{i_{\sigma(k+1)}}} \frac{\partial f}{\partial x_{\sigma(2)}}.$$

We now let $\widehat{\sigma}\{1, \dots, k+1\} \to \{1, \dots, k+1\}$ be the permutation defined by $\widehat{\sigma}(1) = \sigma(2)$, $\widehat{\sigma}(2) = k+1$, $\widehat{\sigma}(k+1) = \sigma(2) = k+1$, and $\widehat{\sigma}(j) = \sigma(j)$ for $j = 3, \dots, k$.

In both cases we have

$$\frac{\partial^{k+1} \boldsymbol{f}}{\partial x_{i_{\sigma(k+1)}} \dots \partial x_{i_{\sigma(1)}}} = \frac{\partial^{k+1} \boldsymbol{f}}{\partial x_{i_{\widehat{\sigma}(k+1)}} \dots \partial x_{i_{\widehat{\sigma}(1)}}} = \frac{\partial^{k+1}}{\partial x_{i_{k+1}}} \frac{\partial^k \boldsymbol{f}}{\partial x_{i_{\widehat{\sigma}(k)}} \dots \partial x_{i_{\widehat{\sigma}(1)}}}$$

using the induction hypothesis

$$= \frac{\partial}{\partial x_{i_{k+1}}} \frac{\partial^k \boldsymbol{f}}{\partial x_{i_k} \dots \partial x_{i_1}} = \frac{\partial^{k+1} \boldsymbol{f}}{\partial x_{i_{k+1}} \dots \partial x_{i_1}}.$$ □

If $m = 1$ then we have a function $f : U \to \mathbb{R}$ and in the case of $k = 2$ we can form an $n \times n$ matrix consisting of the second partial derivatives

$$\boldsymbol{H}(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial^2 f(\boldsymbol{x})}{\partial x_1^2} & \frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 f(\boldsymbol{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\boldsymbol{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\boldsymbol{x})}{\partial x_2^2} & \dots & \frac{\partial^2 f(\boldsymbol{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\boldsymbol{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\boldsymbol{x})}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 f(\boldsymbol{x})}{\partial x_n^2} \end{pmatrix},$$

called the *Hessian*, *Hessian matrix*, or *Hesse matrix*. By Theorem 3.33 the Hessian is symmetric: $\boldsymbol{H}^T = \boldsymbol{H}$. For such a C^2 function we have

$$\begin{aligned} f(\boldsymbol{x}+\boldsymbol{h}) &= f(\boldsymbol{x}) + \mathrm{d}f_{\boldsymbol{x}}(\boldsymbol{h}) + \frac{1}{2}\boldsymbol{h}^T\boldsymbol{H}(\boldsymbol{x})\boldsymbol{h} + \epsilon(\boldsymbol{h})|\boldsymbol{h}|^2 \\ &= f(\boldsymbol{x}) + \sum_{i=1}^{n}\frac{\partial f(\boldsymbol{x})}{\partial x_i}h_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j}h_i h_j + \epsilon(\boldsymbol{h})|\boldsymbol{h}|^2 , \end{aligned} \tag{3.17}$$

where $\epsilon(\boldsymbol{h}) \to 0$ for $\boldsymbol{h} \to \boldsymbol{0}$. The general form of Taylor's theorem with reminder is

Theorem 3.35 (Taylor's theorem with reminder)**.** *If $U \subseteq \mathbb{R}^n$ is an open set, $f : U \to \mathbb{R}$ is a C^k function, $\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{h} \in U$, and $\boldsymbol{h} = (h_1, \dots, h_n)$ then there exist $\xi \in]0,1[$ such that*

$$\begin{aligned} f(\boldsymbol{x}+\boldsymbol{h}) = f(\boldsymbol{x}) &+ \sum_{i=1}^{n}\frac{\partial f(\boldsymbol{x})}{\partial x_i}h_i + \frac{1}{2}\sum_{i_1,i_2=1}^{n}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_{i_1}\partial x_{i_2}}h_{i_1}h_{i_2} + \dots \\ &+ \frac{1}{(k-1)!}\sum_{i_1,i_2,\dots,i_{k-1}=1}^{n}\frac{\partial^{k-1} f(\boldsymbol{x})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_{k-1}}}h_{i_1}h_{i_2}\dots h_{i_{k-1}} \\ &+ \frac{1}{k!}\sum_{i_1,i_2,\dots,i_k=1}^{n}\frac{\partial^{k} f(\boldsymbol{x}+\xi\boldsymbol{h})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_k}}h_{i_1}h_{i_2}\dots h_{i_k} . \end{aligned} \tag{3.18}$$

Proof. Consider the function $g : [0,1] \to \mathbb{R}$ given by $g(t) = f(\boldsymbol{x}+t\boldsymbol{h})$. By repeated use of Example 3.11 we have

$$\frac{\mathrm{d}^\ell g}{\mathrm{d}t^\ell}(t) = \sum_{i_1,i_2,\dots,i_\ell=1}^{n}\frac{\partial^\ell f(\boldsymbol{x}+t\boldsymbol{h})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_\ell}}h_{i_1}h_{i_2}\dots h_{i_\ell} . \tag{3.19}$$

So Theorem 2.38 yields

$$f(\boldsymbol{x}+t\boldsymbol{h}) = g(t) = \sum_{\ell=0}^{k-1}\frac{g^{(\ell)}(0)}{\ell!}t^\ell + \frac{g^{(\ell)}(\xi)}{\ell!}t^\ell , \tag{3.20}$$

for some $\xi \in]0,t[$. Substituting (3.19) into (3.20), and letting $t = 1$ yields (3.18). □

The ϵ-form of Taylor's theorem now follows:

Theorem 3.36 (Taylor's theorem)**.** *If $U \subseteq \mathbb{R}^n$ is an open set, $f : U \to \mathbb{R}$ is a C^k function, $\boldsymbol{x}+t\boldsymbol{h} \in U$ for all $t \in [0,1]$, and $\boldsymbol{h} = (h_1, \dots, h_n)$ then*

$$\begin{aligned} f(\boldsymbol{x}+\boldsymbol{h}) = f(\boldsymbol{x}) &+ \sum_{i=1}^{n}\frac{\partial f(\boldsymbol{x})}{\partial x_i}h_i + \frac{1}{2}\sum_{i_1,i_2=1}^{n}\frac{\partial^2 f(\boldsymbol{x})}{\partial x_{i_1}\partial x_{i_2}}h_{i_1}h_{i_2} + \dots \\ &+ \frac{1}{k!}\sum_{i_1,i_2,\dots,i_k=1}^{n}\frac{\partial^{k} f(\boldsymbol{x})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_k}}h_{i_1}h_{i_2}\dots h_{i_k} + \epsilon(\boldsymbol{h})|\boldsymbol{h}|^k , \end{aligned} \tag{3.21}$$

where $\epsilon(\boldsymbol{h}) \to 0$ for $\boldsymbol{h} \to \boldsymbol{0}$.

Proof. Theorem 3.35 yields

$$f(\boldsymbol{x}+\boldsymbol{h}) = f(\boldsymbol{x}) + \sum_{\ell=1}^{k} \frac{1}{\ell!} \sum_{i_1,\dots,i_\ell=1}^{n} \frac{\partial^\ell f(\boldsymbol{x})}{\partial x_{i_1}\dots\partial x_{i_\ell}} h_{i_1}h_{i_2}\dots h_{i_k}$$
$$+ \frac{1}{k!} \sum_{i_1,i_2,\dots,i_k=1}^{n} \left(\frac{\partial^k f(\boldsymbol{x}+\xi\boldsymbol{h})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_k}} - \frac{\partial^k f(\boldsymbol{x})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_k}} \right) h_{i_1}h_{i_2}\dots h_{i_k} .$$

We now put

$$\epsilon(\boldsymbol{h}) \quad = \quad \frac{1}{k!} \sum_{i_1,i_2,\dots,i_k=1}^{n} \left(\frac{\partial^k f(\boldsymbol{x}+\xi\boldsymbol{h})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_k}} - \frac{\partial^k f(\boldsymbol{x})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_k}} \right) \frac{h_{i_1}h_{i_2}\dots h_{i_k}}{\|\boldsymbol{h}\|^k} ,$$

and we need to show that $\epsilon(\boldsymbol{h}) \to 0$ for $\boldsymbol{h} \to \boldsymbol{0}$. We have $\frac{|h_i|}{\|\boldsymbol{h}\|} \leq 1$ so

$$|\epsilon(\boldsymbol{h})| \leq \frac{1}{k!} \sum_{i_1,i_2,\dots,i_k=1}^{n} \left| \frac{\partial^k f(\boldsymbol{x}+\xi\boldsymbol{h})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_k}} - \frac{\partial^k f(\boldsymbol{x})}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{i_k}} \right| .$$

The continuity of all partial derivatives of order n implies that all terms in the sum goes to zero for $\boldsymbol{h} \to \boldsymbol{0}$ and hence the same is true for $\epsilon(\boldsymbol{h})$. □

Remark 3.37. If we have a C^k vector function $\boldsymbol{f} : U \to \mathbb{R}^m$ then we can use the theorem on each coordinate function f_ℓ of $\boldsymbol{f} = (f_1, \dots, f_m)$.

Just as in the 1-dimensional case we can use the Taylor's theorem to determine (some of) the local maxima and minima.

Theorem 3.38. *Let $U \subseteq \mathbb{R}^n$ be an open set, let $\boldsymbol{x}_0 \in U$, and let $f : I \to \mathbb{R}$ be a C^2-function. If $f(\boldsymbol{x}_0)$ is a local maximum or minimum then $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$.*

Conversely, if $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$ and all eigenvalues of the Hessian $\boldsymbol{H}(\boldsymbol{x}_0)$ are positive then $f(\boldsymbol{x}_0)$ is a local minimum. If $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$ and all eigenvalues of the Hessian $\boldsymbol{H}(\boldsymbol{x}_0)$ are negative then $f(\boldsymbol{x}_0)$ is a local maximum.

Proof. Put $g(t) = f(\boldsymbol{x}_0 + t\nabla f(\boldsymbol{h}))$ then g is differentiable, $g(0) = f(\boldsymbol{x}_0)$, and $g'(0) = \partial_{\nabla f(\boldsymbol{x}_0)} f = \langle \nabla f(\boldsymbol{x}_0), \nabla f(\boldsymbol{x}_0) \rangle = \|\nabla f(\boldsymbol{x}_0)\|^2$. If $f(\boldsymbol{x}_0)$ is1 a local maximum or minimum in $\boldsymbol{x}_0$ then so is $g(0)$. By Lemma 2.25 $g'(0) = 0$, i.e., $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$.

Now suppose $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$ and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the Hessian $\boldsymbol{H}$. If $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ is a corresponding orthonormal basis of eigenvectors of $\boldsymbol{H}$ and $\boldsymbol{h} = h_1\boldsymbol{e}_1 + \dots + h_n\boldsymbol{e}_n$ then we have

$$\begin{aligned} f(\boldsymbol{x}_0+\boldsymbol{h}) &= f(\boldsymbol{x}_0) + \frac{1}{2}\boldsymbol{h}^T\boldsymbol{H}\boldsymbol{h} + \epsilon(\boldsymbol{h})\|\boldsymbol{h}\|^2 \\ &= f(\boldsymbol{x}_0) + \frac{1}{2}(\lambda_1 h_1^2 + \dots + \lambda_n h_n^2) + \epsilon(\boldsymbol{h})\|\boldsymbol{h}\|^2 \\ &= f(\boldsymbol{x}_0) + \frac{1}{2}(\lambda_1 h_1^2 + \dots + \lambda_n h_n^2) + \epsilon(\boldsymbol{h})(h_1^2 + \dots + h_n^2) \\ &= f(\boldsymbol{x}_0) + \frac{1}{2}((\lambda_1 + 2\epsilon(\boldsymbol{h}))h_1^2 + \dots + (\lambda_n + 2\epsilon(\boldsymbol{h}))h_n^2) . \end{aligned}$$

We have $\epsilon(\boldsymbol{h}) \to \boldsymbol{0}$ for $\boldsymbol{h} \to \boldsymbol{0}$ so if $\lambda_1, \ldots, \lambda_n \neq 0$ then we can find $r > 0$ such that $|2\epsilon(\boldsymbol{h})| < \min\{|\lambda_1|, \ldots, |\lambda_n|\}$ for $\|\boldsymbol{h}\| < r$. In that case $\lambda_n + 2\epsilon(\boldsymbol{h})$ has the same sign as λ_k. So if all λ_k are negative $f(\boldsymbol{x}_0 + \boldsymbol{h}) < f(\boldsymbol{x}_0)$ for $\|\boldsymbol{h}\| < r$ and we have a local maximum and if they all are positive $f(\boldsymbol{x}_0 + \boldsymbol{h}) > f(\boldsymbol{x}_0)$ for $\|\boldsymbol{h}\| < r$ and we have a local minimum. □

Remark 3.39. If $\nabla f(\boldsymbol{x}_0) = \boldsymbol{0}$ and we have at least one positive and one negative eigenvalue of the Hessian then f increases in some direction and decreases in some other directions. We then say that $f(\boldsymbol{x}_0)$ is a *saddle point*, and we know for certain that we neither have a local maximum nor a local minimum.

3.4 Curves and line integrals

If we have a differential vector function $\boldsymbol{x} : [a, b] \to \mathbb{R}^n$ then the image is a *curve* in $\mathbb{R}^n$ and we call $\boldsymbol{x}$ a *parametrisation* of the curve. If $n = 2$ we have a planar curve and if $n = 3$ we have a space curve.

Example 3.12. Consider the quarter circle $x^2 + y^2 = 1$ and $x, y \geq 0$ in the plane. The vector function $\boldsymbol{x} : [0, \frac{\pi}{2}] \to \mathbb{R}^2$ given $\boldsymbol{x}(t) = (\cos t, \sin t)$ is a parametrisation of the quarter circle, but so is $\boldsymbol{y} : [0, 1] \to \mathbb{R}^2$ given by $\boldsymbol{y}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$. Indeed, $\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 = \frac{1-2t^2+t^4+4t^2}{1+2t^2+t^4} = \frac{1+2t^2+t^4}{1+2t^2+t^4} = 1$. So a curve can be parameterised in many ways.

The variable $t \in [a, b]$ may represent time and the curve the trajectory of a particle moving through space. With such an interpretation the derivative $\boldsymbol{x}'(t)$ is the *velocity* at time t and the length $\|\boldsymbol{x}'(t)\|$ is the *speed*. This is often expressed as $\frac{\mathrm{d}s}{\mathrm{d}t} = \left\|\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}\right\|$, where s represent *arc-length* on the curve. In order to find the length of the curve, i.e., how long have the particle moved we need to integrate the speed:

$$L = \int_a^b \frac{\mathrm{d}s}{\mathrm{d}t}\,\mathrm{d}t = \int_a^b \|\boldsymbol{x}'(t)\|\,\mathrm{d}t\,. \tag{3.22}$$

An other way to determine the length of the curve is to approximate by polygons. We can evaluate the curve in points $a = t_0 < t_1, < \cdots < t_n = b$ and consider the polygon with vertices $\boldsymbol{x}(t_0), \boldsymbol{x}(t_1), \ldots, \boldsymbol{x}(t_n)$, see Figure 3.4. The length of the

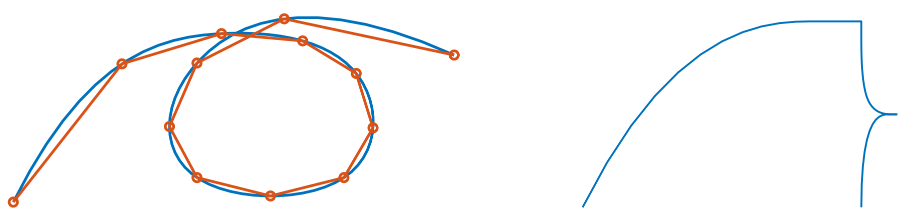

Figure 3.4: To the left a curve and a polygonal approximation. To the right a curve with a kink and a cusp.

polygon is obviously $\sum_{k=1}^{N} \|\boldsymbol{x}(t_k) - \boldsymbol{x}(t_{k-1}\|$. If $h_k = t_k - t_{k-1}$ then Taylor's theorem tells us that $\|\boldsymbol{x}(t_k) - \boldsymbol{x}(t_{k-1})\| = \|\boldsymbol{x}'(t_{k-1})\| h_k + \epsilon(h_k)|h_k|$ and using this it can be shown that if $\max_{k=1,\dots,n} |h_k| \to 0$ then the polygon length converges to the length defined by (3.22).

If $\boldsymbol{x}'(x) \neq \boldsymbol{0}$ for all $x \in [a, b]$ then the curve has a well defined *tangent* at each point in the direction of $\boldsymbol{x}'(x)$. We then call the curve *regular*. At points where $\boldsymbol{x}'(x) = \boldsymbol{0}$ the curve may have a kink or a cusp, see Figure 3.4.

Now suppose that we have a domain $U \subseteq \mathbb{R}^n$ with $\boldsymbol{x}([a, b]) \subseteq U$ and a function $f : U \to \mathbb{R}$. We want to integrate f over the curve. If we want to mimic the definition of the Riemann integral then we could cut the curve into pieces, choose a point in each piece, evaluate the function f in that point, multiply the value of the function with the length of the piece, and add the result of all the pieces. If we compare the length of piece between $\boldsymbol{x}(t)$ and $\boldsymbol{x}(t + h)$ with h then we have $\frac{\|\boldsymbol{x}(t+h) - \boldsymbol{x}(t)\|}{|h|} = \|\boldsymbol{x}'(t)\| + \epsilon(h)$. So if the curve is sampled more and more densely the procedure outline above leads to the expression

$$\int_0^L f(\boldsymbol{x}) \,\mathrm{d}s = \int_a^b f(\boldsymbol{x}(t)) \frac{\mathrm{d}s}{\mathrm{d}t} \,\mathrm{d}t = \int_a^b f(\boldsymbol{x}(t)) \|\boldsymbol{x}'(t)\| \,\mathrm{d}t \,, \tag{3.23}$$

called the *line integral* of f along the curve. We will not prove that, but we will prove that all parametrisations yields the same result.

Suppose we have a differentiable function $g : [c, d] \to [a, b]$. Then Theorem 2.63 tells us that

$$\int_a^b f(\boldsymbol{x}(t)) \|\boldsymbol{x}'(t)\| \,\mathrm{d}t = \int_c^d f(\boldsymbol{x}(g(u))) \|\boldsymbol{x}'(g(u))\| g'(u) \,\mathrm{d}u \,.$$

On the other hand if g is monotonically increasing then $\boldsymbol{y} = \boldsymbol{x} \circ g$ is also a parametrisation and $\boldsymbol{y}'(u) = \boldsymbol{x}'(g(u)) g'(u)$. Hence

$$\int_c^d f(\boldsymbol{y}(u)) \|\boldsymbol{y}'(u)\| \,\mathrm{d}u = \int_c^d f(\boldsymbol{x}(g(u))) \|\boldsymbol{x}'(g(u))\| g'(u) \,\mathrm{d}u \,,$$

and we see that the expression (3.23) does not depend on the parametrisation.

If $\boldsymbol{v} : U \to \mathbb{R}^n$ is a vector function on domain $U \subseteq \mathbb{R}^n$ and $\boldsymbol{x}([a, b]) \subseteq U$ then we can take the inner product with the tangent vector $\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \frac{\boldsymbol{x}'}{\|\boldsymbol{x}'\|}$ and integrate the result:

$$\begin{aligned} \int_0^L \langle \boldsymbol{t}, \boldsymbol{v} \rangle \,\mathrm{d}s &= \int_0^L \left\langle \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s}, \boldsymbol{v}(\boldsymbol{x}) \right\rangle \mathrm{d}s = \int_a^b \left\langle \frac{\mathrm{d}t}{\mathrm{d}s} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}, \boldsymbol{v}(\boldsymbol{x}) \right\rangle \frac{\mathrm{d}s}{\mathrm{d}t} \,\mathrm{d}t \\ &= \int_a^b \left\langle \left(\frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}} \right) \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}, \boldsymbol{v}(\boldsymbol{x}) \right\rangle \frac{\mathrm{d}s}{\mathrm{d}t} \,\mathrm{d}t = \int_a^b \langle \boldsymbol{x}'(t), \boldsymbol{v}(\boldsymbol{x}(t)) \rangle \,\mathrm{d}t \,. \end{aligned} \tag{3.24}$$

This is called the line integral of $\boldsymbol{v}$ along the curve. A calculation as above shows that the result does not depend on the parametrisation (Exercise 3.2).

3.5 Integration in higher dimensions

Integration in $\mathbb{R}^n$ for $n \geq 2$ is considerable more involved that in integration in $\mathbb{R}$ as we did in Section 2.3. The length of the interval $[a, b]$ is $b - a$, but if we have a domain in the plane what is the area?

If $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ is continuous then we would like to integrate f over A, i.e., calculate $\int_A f(x_1, \dots, x_n)\, \mathrm{d}x_n \dots \mathrm{d}x_1$. We have on example where it is clear how this should be done. If $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ then we simply let

$$\int_A f(\boldsymbol{x})\, \mathrm{d}x_n \dots \mathrm{d}x_1 = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n)\, \mathrm{d}x_n \dots \mathrm{d}x_1 \,. \tag{3.25}$$

We only need to show the following result.

Lemma 3.40. *If $A = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ is continuous then the function $g : [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \to \mathbb{R}$ given by $g(x_1, \dots, x_{n-1}) = \int_{a_n}^{b_n} f(x_1, \dots, x_{n-1}, t)\, \mathrm{d}t$ is continuous.*

Proof. We have

$$|g(x_1, \dots, x_{n-1}) - g(y_1, \dots, y_{n-1})| \leq \int_{a_n}^{b_n} |f(x_1, \dots, x_{n-1}, t) - f(y_1, \dots, y_{n-1}, t)|\, \mathrm{d}t \,,$$

and as f is uniformly continuous it is not hard to show that so is g. The details is left as Exercise 3.3 □

3.5.1 Integration in the plane

Let $\Omega \subseteq \mathbb{R}^2$ be a domain in the plane and let $f : \Omega \to \mathbb{R}$ be a continuous function. We want to define the integral of f over Ω and we will write it as $\int_\Omega f(\boldsymbol{x})\, \mathrm{d}A$. Suppose we have a differentiable map $\boldsymbol{x} : [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^2$ such that the image of $\boldsymbol{x}$ is Ω, i.e., $\boldsymbol{x}([a_1, b_1] \times [a_2, b_2]) = \Omega$, i.e., a parametrisation of Ω.

Inspired by the definition of a line integral we will first see what happens to the area of a small square under a differentiable map. The differential $\mathrm{d}\boldsymbol{x}$ maps the unit vectors $(1, 0)$ and $(0, 1)$ to the partial derivatives $\frac{\partial \boldsymbol{x}}{\partial u_1}$ and $\frac{\partial \boldsymbol{x}}{\partial u_2}$ and a small square with edges $h\boldsymbol{e}_1$ and $h\boldsymbol{e}_2$ has area h^2 and is approximately mapped to a parallelogram with edges $h\frac{\partial \boldsymbol{x}}{\partial u_1}$ and $h\frac{\partial \boldsymbol{x}}{\partial u_2}$. The area of that parallelogram is $h^2 |\det \boldsymbol{J}|$ where $\boldsymbol{J}$ is the Jacobian matrix. We will now define the integral of f over Ω by

$$\int_\Omega f(\boldsymbol{x})\, \mathrm{d}A = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(\boldsymbol{x}(u, v)) |\det \boldsymbol{J}(\boldsymbol{x}(u_1, u_2))|\, \mathrm{d}u_2\, \mathrm{d}u_1 \,. \tag{3.26}$$

Just as for line integrals it can be shown that the result is independent of the parametrisation.

Example 3.13. Consider the parametrisation $\boldsymbol{x} : [0, r] \times [0, 2\pi] \to D_r$, of the disc with radius r, given by $\boldsymbol{x}(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta)$ (polar coordinates). The Jacobian of $\boldsymbol{x}$ is $\boldsymbol{J} = \left(\frac{\partial \boldsymbol{x}}{\partial \rho}\ \frac{\partial \boldsymbol{x}}{\partial \theta}\right) = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}$ and the determinant is $\det \boldsymbol{J} = \rho$. The area of the disc can be found by integrating the constant 1 over the disc, i.e.,

$$\int_{D_r} 1 \,\mathrm{d}A = \int_0^r \int_0^{2\pi} \det \boldsymbol{J} \,\mathrm{d}\theta \,\mathrm{d}\rho = \int_0^r \int_0^{2\pi} \rho \,\mathrm{d}\theta \,\mathrm{d}\rho = \int_0^r 2\pi\rho \,\mathrm{d}\rho = \pi r^2 \,.$$

3.5.2 Integration in space and higher dimensions

Domains in higher dimensions can be treated the same way. Let $\Omega \subseteq \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be a continuous function. We want to define the integral of f over Ω and we will write it as $\int_\Omega f(\boldsymbol{x}) \,\mathrm{d}V$. Suppose we have a differentiable map $\boldsymbol{x} : [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \to \mathbb{R}^n$ such that the image of $\boldsymbol{x}$ is Ω, i.e., $\boldsymbol{x}([a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]) = \Omega$, i.e., a parametrisation of Ω. We then define the integral of f over Ω by

$$\int_\Omega f(\boldsymbol{x}) \,\mathrm{d}V = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} f(\boldsymbol{x}(u, v)) |\det \boldsymbol{J}(\boldsymbol{x}(u_1, u_2))| \,\mathrm{d}u_n \,\ldots \,\mathrm{d}u_2 \,\mathrm{d}u_1 \,. \quad (3.27)$$

Again it can be shown that the result is independent of the parametrisation.

Example 3.14. Consider the following parametrisation of the solid ball with radius r: $\boldsymbol{x} : [0, r] \times [0, \pi] \times [0, 2\pi] \to B_r$ given by $\boldsymbol{x}(\rho, \theta, \phi) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. The Jacobian is

$$\boldsymbol{J} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} ,$$

with determinant

$$\begin{aligned} \det \boldsymbol{J} = \rho^2 \cos\theta(\cos\theta\sin\theta\cos^2\phi + \cos\theta\sin\theta\sin^2\phi) \\ + \rho^2 \sin\theta(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi) \\ = \rho^2(\cos^2\theta\sin\theta + \sin^3\theta) = \rho^2\sin\theta \,. \end{aligned}$$

So the volume of the ball is

$$\begin{aligned} \int_{B_r} 1 \,\mathrm{d}V &= \int_0^r \int_0^\pi \int_0^{2\pi} \det \boldsymbol{J} \,\mathrm{d}\phi \,\mathrm{d}\theta \,\mathrm{d}\rho \\ &= \int_0^r \int_0^\pi \int_0^{2\pi} \rho^2 \sin\theta \,\mathrm{d}\phi \,\mathrm{d}\theta \,\mathrm{d}\rho = \int_0^r \int_0^\pi 2\pi\rho^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\rho \\ &= \int_0^r 2\pi[-\cos\theta]_0^\pi \rho^2 \,\mathrm{d}\rho = \int_0^r 4\pi\rho^2 \,\mathrm{d}\rho = \frac{4}{3}\pi r^3 \,. \end{aligned}$$

3.5.3 Surface integrals

If $U \subseteq \mathbb{R}^2$ and we have a differentiable vector function $\boldsymbol{x} : U \to \mathbb{R}^n$ with $n \geq 3$ then the image is called a *surface* in $\mathbb{R}^n$ and we call $\boldsymbol{x}$ a *parametrisation* of the surface.

If the the partial derivatives $\frac{\partial \boldsymbol{x}}{\partial u}$ and $\frac{\partial \boldsymbol{x}}{\partial v}$ spans a plane, i.e., they are linearly independent, then we call the surface *regular* and it has a *tangent plane* at each point, spanned by the partial derivatives, see Figure 3.5.

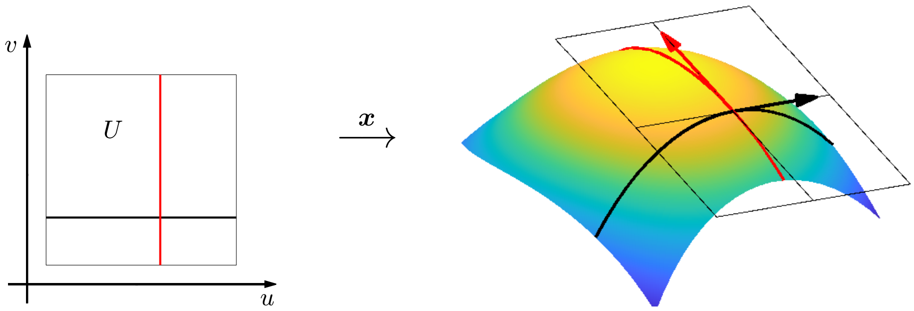

Figure 3.5: A parametrisation of a surface.

We would like to be able to integrate functions over the surface. A small square in the parameter plane with edges $h\boldsymbol{e}_1$ and $h\boldsymbol{e}_2$ are approximately mapped to a parallelogram with edges $h\frac{\partial \boldsymbol{x}}{\partial u}$ and $h\frac{\partial \boldsymbol{x}}{\partial v}$. We first determine the area of that parallelogram. If we let $h\frac{\partial \boldsymbol{x}}{\partial u}$ be the base then it has length $\left\|h\frac{\partial \boldsymbol{x}}{\partial u}\right\|$ and the height of the parallelogram is

$$\left\| h\frac{\partial \boldsymbol{x}}{\partial v} - \frac{\langle h\frac{\partial \boldsymbol{x}}{\partial v}, h\frac{\partial \boldsymbol{x}}{\partial u}\rangle}{\left\|h\frac{\partial \boldsymbol{x}}{\partial u}\right\|^2} h\frac{\partial \boldsymbol{x}}{\partial u} \right\| = |h| \sqrt{\left\|\frac{\partial \boldsymbol{x}}{\partial v}\right\|^2 - \frac{\langle \frac{\partial \boldsymbol{x}}{\partial v}, \frac{\partial \boldsymbol{x}}{\partial u}\rangle^2}{\left\|\frac{\partial \boldsymbol{x}}{\partial u}\right\|^2}}\,.$$

So the area is

$$\begin{aligned} h^2 \sqrt{\left\|\frac{\partial \boldsymbol{x}}{\partial u}\right\|^2 \left\|\frac{\partial \boldsymbol{x}}{\partial v}\right\|^2 - \left\langle \frac{\partial \boldsymbol{x}}{\partial v}, \frac{\partial \boldsymbol{x}}{\partial u}\right\rangle^2} &= h^2 \sqrt{\det \begin{pmatrix} \langle \frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial u}\rangle & \langle \frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v}\rangle \\ \langle \frac{\partial \boldsymbol{x}}{\partial v}, \frac{\partial \boldsymbol{x}}{\partial u}\rangle & \langle \frac{\partial \boldsymbol{x}}{\partial v}, \frac{\partial \boldsymbol{x}}{\partial v}\rangle \end{pmatrix}} \\ &= h^2 \sqrt{\det \left(\begin{pmatrix} \frac{\partial \boldsymbol{x}}{\partial u} \\ \frac{\partial \boldsymbol{x}}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{pmatrix} \right)} = h^2 \sqrt{\det(\boldsymbol{J}^T \boldsymbol{J})}\,, \end{aligned}$$

where $\boldsymbol{J} = \begin{pmatrix} \frac{\partial \boldsymbol{x}}{\partial u} & \frac{\partial \boldsymbol{x}}{\partial v} \end{pmatrix}$ is the Jacobian of $\boldsymbol{x}$. If S is a surface and $\boldsymbol{x} : [a_1, b_1] \times [a_2, b_2] \to S$ is a parametrisation of S then the integral of a function f defined on a set $A \subseteq \mathbb{R}^n$ with $\boldsymbol{x}(U) \subseteq A$ is

$$\int_S f \,\mathrm{d}A = \int_{a_1}^{b_1} \int_{a_1}^{b_1} f(\boldsymbol{x}(u, v)) \,\mathrm{d}v \,\mathrm{d}u\,. \tag{3.28}$$

As before it can be shown that the result does not depend on the parametrisation $\boldsymbol{x}$. If $f = 1$ on S then we obtain the area of S.

Example 3.15. Consider the parametrisation of the sphere with radius r given by $\boldsymbol{x}(\theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$ for $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$. The Jacobian is

$$\boldsymbol{J} = \begin{pmatrix} r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ -r\sin\theta & 0 \end{pmatrix},$$

and we have

$$\begin{aligned} \left\| \frac{\partial \boldsymbol{x}}{\partial u} \right\|^2 &= r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2\,, \\ \left\langle \frac{\partial \boldsymbol{x}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v} \right\rangle &= -r^2\cos\theta\sin\theta\cos\phi\sin\phi + r^2\cos\theta\sin\theta\cos\phi\sin\phi = 0\,, \\ \left\| \frac{\partial \boldsymbol{x}}{\partial v} \right\|^2 &= r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta\,. \end{aligned}$$

So $\boldsymbol{J}^T\boldsymbol{J} = \left(\begin{smallmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{smallmatrix}\right)$, the determinant is $\det(\boldsymbol{J}^T\boldsymbol{J}) = r^4\sin^2\theta$, and the area of the sphere is

$$\begin{aligned} \int_S 1\,\mathrm{d}A = \int_0^\pi \int_0^{2\pi} \sqrt{\det(\boldsymbol{J}^T\boldsymbol{J})}\,\mathrm{d}\phi\,\mathrm{d}\theta &= \int_0^\pi \int_0^{2\pi} r^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \int_0^\pi 2\pi r^2\sin\theta\,\mathrm{d}\theta = 2\pi r^2[-\cos\theta]_0^\pi = 4\pi r^2\,. \end{aligned}$$

3.6 Vector fields

Let $U \subseteq \mathbb{R}^n$ be an open set and let $\boldsymbol{f} : U \to \mathbb{R}^n$. Then we can think of $\boldsymbol{f}$ as a *vector field* on U, i.e., attach the vector $\boldsymbol{f}(\boldsymbol{x})$ to the point $\boldsymbol{x}$, see Figure 3.6 left. It could

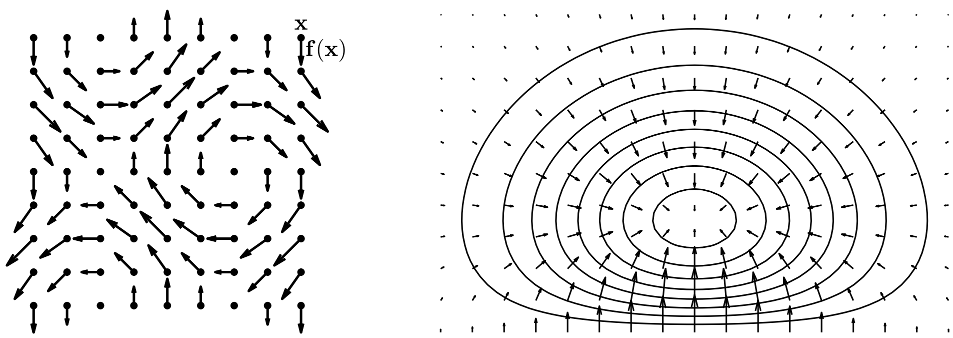

Figure 3.6: Left: A vector field on $\mathbb{R}^2$. Right: Level sets of a function and its gradient field.

be the velocity field of a fluid, the inner forces or strains in a beam, the electric field around an antenna, etc.

Definition 3.41 (Gradient)**.** Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}$ be differentiable. If $\langle \cdot, \cdot \rangle$ is an inner product then the *gradient* of f at $\in U$ is a vector $\nabla f(\boldsymbol{x}) \in \mathbb{R}^n$ such that $\langle \nabla f(\boldsymbol{x}), \boldsymbol{h} \rangle = \mathrm{d}f_{\boldsymbol{x}}\boldsymbol{h}$ for all $\boldsymbol{h} \in \mathbb{R}^n$. If we consider $\boldsymbol{x} \in \mathbb{R}^n$ as a column matrix, the inner product $\langle \cdot, \cdot \rangle$ is the usual inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T \boldsymbol{y}$, and $\boldsymbol{J}(\boldsymbol{x}) \in \mathbb{R}^{1\times n}$ is the Jacobian matrix then we have $\nabla f(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{x})^T$.

Remark 3.42. The gradient $\nabla f(\boldsymbol{x})$ of a function $f : U \to \mathbb{R}$ is often confused with the differential $\mathrm{d}f_{\boldsymbol{x}}$. But $\mathrm{d}f_{\boldsymbol{x}}$ is a linear map $\mathbb{R}^n \to \mathbb{R}$, i.e., a linear form and is defined with out any reference to an inner product. We can also define $\mathrm{d}f_{\boldsymbol{x}}$ if we replace $\mathbb{R}^n$ with a abstract vector space. The gradient is a vector $\nabla f(\boldsymbol{x}) \in \mathbb{R}^n$ and can only be defined if we have an inner product. Different inner products leads to different gradients. If we consider vectors in $\mathbb{R}^n$ as columns and use matrix notation then $\mathrm{d}f_{\boldsymbol{x}}$ is represented by the Jacobi matrix $\boldsymbol{J}(\boldsymbol{x}) \in \mathbb{R}^{1\times n}$ and $\nabla f(\boldsymbol{x}) \in \mathbb{R}^{n\times 1}$. It is only if we use the usual inner product in $\mathbb{R}^n$ that $\nabla f(\boldsymbol{x}) = \boldsymbol{J}(\boldsymbol{x})^T$.

Remark 3.43. If f is a differentiable function with gradient ∇f and $\boldsymbol{v} \in \mathbb{R}^n$ is a vector that is tangent to a level set. Then

$$\langle \nabla f(\boldsymbol{x}), \boldsymbol{v} \rangle = \mathrm{d}f_{\boldsymbol{x}}(\boldsymbol{v}) = \partial_{\boldsymbol{v}} f(\boldsymbol{x}) = 0\,.$$

That is, the gradient is orthogonal to the level sets, see Figure 3.6 right.

In dimension one a vector field on an interval I is just a function $f : I \to \mathbb{R}$ and by integrating it we see that it is gradient field: $f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_{x_0}^{x} f(t)\,\mathrm{d}t$. In higher dimensions gradient fields are special.

Let $U \subseteq \mathbb{R}^n$ be open and let $\boldsymbol{v}(\boldsymbol{x}) = (v_1(\boldsymbol{x}), \dots, v_n(\boldsymbol{x}))$ be a C^1 vector field (or function) $\mathbb{R}^n \to \mathbb{R}^n$. If $\boldsymbol{v}$ is a gradient field, $\boldsymbol{v} = \nabla f$, then $v_k = \frac{\partial f}{\partial x_k}$ and $\boldsymbol{v}$ has the Jacobian matrix

$$\boldsymbol{J} = \begin{pmatrix} \nabla v_1 \\ \vdots \\ \nabla v_n \end{pmatrix} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix},$$

i.e., $\boldsymbol{J} = \boldsymbol{H}$ the Hessian of f. In that case we can see that $\boldsymbol{J}$ is symmetric. The converse is true under some condition on the domain U.

Theorem 3.44. *Let $B(\boldsymbol{x}_0, r) \subseteq \mathbb{R}^n$ be an open ball. A C^1 vector field $\boldsymbol{v} : B(\boldsymbol{x}_0, r) \to \mathbb{R}^n$ is a gradient field if and only if the Jacobian matrix $\boldsymbol{J}$ is symmetric at all points $\boldsymbol{x} \in U$. In that case we have $\boldsymbol{v} = \mathrm{d}f$, where*

$$f(\boldsymbol{x}) = \int_0^1 \langle \boldsymbol{v}(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0)), \boldsymbol{x} - \boldsymbol{x}_0 \rangle\,\mathrm{d}t\,. \tag{3.29}$$

Proof. If $\boldsymbol{v} = \mathrm{d}f$ then the Jacobian is $\boldsymbol{J} = \boldsymbol{H}$ is the Hessian of f which is symmetric. Conversely, assume the Jacobian $\boldsymbol{J} = \left(\frac{\partial v_i}{\partial x_j}\right)$ is symmetric and $\|\boldsymbol{h}\| < r$. Put

$$g(s, \boldsymbol{h}) = \int_0^s \langle \boldsymbol{v}(\boldsymbol{x}_0 + t\boldsymbol{h}), \boldsymbol{h} \rangle\,\mathrm{d}t\,.$$

Observe that $f(\boldsymbol{x}) = g(1, \boldsymbol{x} - \boldsymbol{x}_0)$. We have $g(0, \boldsymbol{h}) = 0$ for all $\boldsymbol{h}$, so $\frac{\partial g}{\partial h_i}(0, \boldsymbol{h}) = 0$ for all $i = 1, \ldots, n$. Furthermore,

$$\begin{aligned}\frac{\partial^2 g}{\partial t \partial h_i}(s, \boldsymbol{h}) = \frac{\partial^2 g}{\partial h_i \partial t}(s, \boldsymbol{h}) &= \frac{\partial}{\partial h_i}\langle \boldsymbol{v}(\boldsymbol{x}_0 + t\boldsymbol{h}), \boldsymbol{h}\rangle \\ &= \frac{\partial}{\partial h_i} \sum_{j=1}^{n} v_j(\boldsymbol{x}_0 + t\boldsymbol{h}) h_j \\ &= \sum_{j=1}^{n} t \frac{\partial v_j}{\partial h_i}(\boldsymbol{x}_0 + t\boldsymbol{h}) h_j + v_i(\boldsymbol{x}_0 + t\boldsymbol{h})\end{aligned}$$

using the symmetri of $\boldsymbol{J}$

$$\begin{aligned}&= t \sum_{j=1}^{n} \frac{\partial v_i}{\partial h_j}(\boldsymbol{x}_0 + t\boldsymbol{h}) h_j + v_i(\boldsymbol{x}_0 + t\boldsymbol{h}) \\ &= t\langle \nabla v_i(\boldsymbol{x}_0 + t\boldsymbol{h}), \boldsymbol{h}\rangle + v_i(\boldsymbol{x}_0 + t\boldsymbol{h}) \\ &= \frac{\partial}{\partial t}(t v_i(\boldsymbol{x}_0 + t\boldsymbol{h}))\,.\end{aligned}$$

As $\frac{\partial g}{\partial h_i}(t, \boldsymbol{h})$ and $t v_i(\boldsymbol{x}_0 + t\boldsymbol{h})$ both are zero for $t = 0$ and they have the same derivative with respect to t we must have $\frac{\partial g}{\partial h_i}(t, \boldsymbol{h}) = v_i(\boldsymbol{x}_0 + t\boldsymbol{h})$. Letting $t = 1$ we see that $\nabla f(\boldsymbol{x}) = \nabla g(1, \boldsymbol{x} - \boldsymbol{x}_0) = \boldsymbol{v}(\boldsymbol{x})$. □

Remark 3.45. The theorem says that a vector field with a symmetric Jacobian matrix locally is a gradient field. We may or may not be able to piece local solutions together and obtain a global solution, see Figure 3.7.

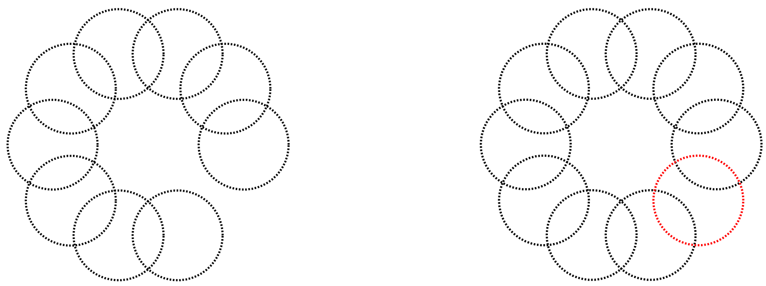

Figure 3.7: If we have a vector field $\boldsymbol{v}$ with symmetric Jacobian matrix and functions f_i with $\nabla f_i = \boldsymbol{v}$ on each open ball, then they differ by a constant on the overlaps. To the left we can add constants to the functions and obtain a function on the union of the balls. But if we add the red ball to the right we may have different constants on the two overlaps with the red ball and in that case we cannot find a solution on the union of the balls.

Example 3.16. Let $\boldsymbol{v} = \frac{(x,y)}{10\sqrt{x^2+y^2}}$ defined on the annulus $\frac{1}{4} < x^2 + y^2 < 1$, see Figure 3.8 left. It is the gradient of the function given by $f(x,y) = \frac{\sqrt{x^2+y^2}}{10}$, shown below.

Example 3.17. Let $\boldsymbol{v} = \frac{(-y,x)}{10\sqrt{x^2+y^2}}$ defined on the annulus $\frac{1}{4} < x^2 + y^2 < 1$, see Figure 3.8 right. It is the gradient of the function given by $f(x,y) = \frac{\arctan \frac{y}{x}}{10}$, shown below. But we can not define it on all of the annulus.

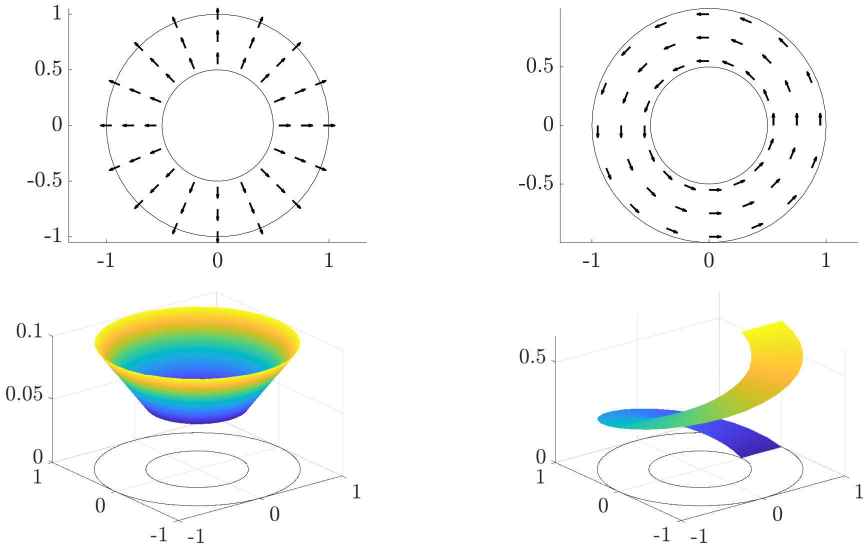

Figure 3.8: Top two vector fields. Below functions having these gradients.

Remark 3.46. To characterise the domains $U \in \mathbb{R}^n$ where symmetry of the Jacobian matrix of a vector field $\boldsymbol{v}$ on U implies the existence of a function $f : U \to \mathbb{R}$ with $\nabla f = \boldsymbol{v}$ we need concepts from algebraic topology: The domain should be simply connected, but to explain the precise meaning of that is beyond the scope of this book.

Definition 3.47 (Divergence)**.** Let $U \subseteq \mathbb{R}^n$ be an open set and let $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be a differentiable vector field on U, i.e., $\boldsymbol{v} : U \to \mathbb{R}^n$ is differentiable. The *divergence* of $\boldsymbol{v}$ is

$$\operatorname{div} \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \cdots + \frac{\partial v_n}{\partial x_n}. \tag{3.30}$$

The notation $\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v}$ is often seen and if we consider ∇ as a vector with components $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ and formally calculates the inner product between ∇ and $\boldsymbol{v}$ then we obtain the divergence, c.f. Exercise 3.1.

Remark 3.48. As the Jacobian matrix of $\boldsymbol{v}$ is

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_n}{\partial x_1} & \cdots & \frac{\partial v_n}{\partial x_n} \end{pmatrix},$$

we have $\operatorname{div} \boldsymbol{v} = \operatorname{trace} \boldsymbol{J}$.

If we think of the vector field as the velocity field of a fluid then the divergence measures the infinitesimal expansion or contraction of the fluid . The velocity of an incompressible fluid has zero divergence.

Example 3.18. If $\boldsymbol{v} = \frac{(x,y)}{10\sqrt{x^2+y^2}}$ from Example 3.16 then

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= \frac{\partial}{\partial x} \frac{x}{10\sqrt{x^2+y^2}} + \frac{\partial}{\partial y} \frac{y}{10\sqrt{x^2+y^2}} \\
&= \frac{1}{10}\left(\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{(x^2+y^2)^{3/2}} \right) \\
&= \frac{1}{10}\left(\frac{x^2+y^2}{(x^2+y^2)^{3/2}} - \frac{x^2}{(x^2+y^2)^{3/2}} + \frac{x^2+y^2}{(x^2+y^2)^{3/2}} - \frac{y^2}{(x^2+y^2)^{3/2}} \right) \\
&= \frac{1}{10} \frac{y^2+x^2}{(x^2+y^2)^{3/2}} = \frac{1}{10\sqrt{x^2+y^2}} \,.
\end{aligned}
$$

Example 3.19. If $\boldsymbol{v} = \frac{(-y,x)}{10\sqrt{x^2+y^2}}$ from Example 3.17 then

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= \frac{\partial}{\partial x} \frac{-y}{10\sqrt{x^2+y^2}} + \frac{\partial}{\partial y} \frac{x}{10\sqrt{x^2+y^2}} \\
&= \frac{1}{10}\left(\frac{-yx}{(x^2+y^2)^{3/2}} + \frac{xy}{(x^2+y^2)^{3/2}} \right) = 0 \,.
\end{aligned}
$$

Definition 3.49. Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be a C^2 function. The *Laplacian* of f is

$$
\triangle f = \operatorname{div} \nabla f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \,. \tag{3.31}
$$

The operator $\triangle = \operatorname{div} \nabla$ is called the *Laplace operator*.

3.6.1 The divergence theorem

Given a vector field on a domain. The divergence theorem says that the integral of the divergence over the domain is the same as the integral of the normal component of the vector field over the boundary of the domain.

Dimension 1

Consider a differentiable vector field on an open interval $I \subseteq \mathbb{R}$, i.e., a differentiable function $f : I \to \mathbb{R}$. and an interval $[a, b] \subseteq I$, see Figure 3.9. We can think of f as the velocity of electric charges in a wire, as the velocity of a gas in a thin tube, or

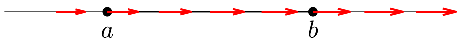

Figure 3.9: Flux out of an interval.

as the velocity of cars on straight road. We are interested in *flux* of the vector field f, i.e., how much charge or air that flows in or out of the interval $[a, b]$. The rate of flow out of the interval at a is $-f(a)$ and the rate of flow out of the interval at b is $f(b)$. So the total rate of flow (or flux) out of the interval is

$$f(b) - f(a) = \int_a^b f'(x)\,\mathrm{d}x\,. \tag{3.32}$$

Dimension 2

We now consider a C^1 vector field $\boldsymbol{v}(x, y) = v_1(x, y)\boldsymbol{e}_1 + v_2(x, y)\boldsymbol{e}_2$ on an open set $U \subseteq \mathbb{R}^2$ and at first a rectangle $[a, b] \times [c, d] \subseteq U$, see Figure 3.10. The rates of flow

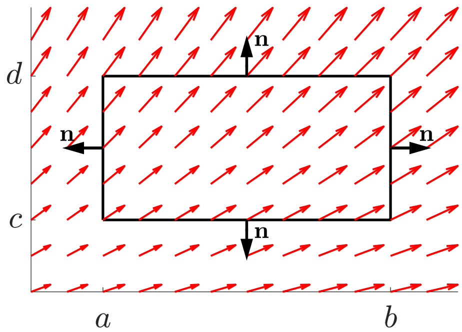

Figure 3.10: Flux out of a rectangle.

out of the rectangle at the sides $x = a$ or $x = b$ and $c \le y \le d$ are

$$-\int_c^d v_1(a, y)\,\mathrm{d}y = \int_c^d \langle \boldsymbol{v}(a, y), -\boldsymbol{e}_1\rangle = \int_c^d \langle \boldsymbol{v}(a, y), \boldsymbol{n}(a, y)\rangle\,,$$

and

$$\int_c^d v_1(b, y) = \int_c^d \langle \boldsymbol{v}(b, y), \boldsymbol{e}_1\rangle\,\mathrm{d}y = \int_c^d \langle \boldsymbol{v}(b, y), \boldsymbol{n}(b, y)\rangle\,\mathrm{d}y\,,$$

respectively, where $\boldsymbol{n}$ is the outward normal. So the total flux out of the two vertical sides is

$$\begin{aligned}\int_c^d \langle \boldsymbol{v}(b, y), \boldsymbol{n}(b, y)\rangle\,\mathrm{d}y + \int_c^d \langle \boldsymbol{v}(a, y), \boldsymbol{n}(a, y)\rangle\,\mathrm{d}y \\ = \int_c^d v_1(b, y)\,\mathrm{d}y - \int_c^d v_1(a, y)\,\mathrm{d}y = \int_c^d (v_1(b, y) - v_1(a, y))\,\mathrm{d}y \\ = \int_c^d \int_a^b \frac{\partial v_1}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,.\end{aligned}$$

Similar, the total flux out of the two horizontal sides is

$$\int_a^b \langle \boldsymbol{v}(x,d), \boldsymbol{n}(x,d)\rangle \,\mathrm{d}x + \int_a^b \langle \boldsymbol{v}(x,c), \boldsymbol{n}(x,c)\rangle \,\mathrm{d}x = \int_a^b v_2(x,d)\,\mathrm{d}x - \int_a^b v_2(x,c)\,\mathrm{d}x$$
$$= \int_a^b (v_2(x,d) - v_2(x,c))\,\mathrm{d}x = \int_a^b \int_c^d \frac{\partial v_2}{\partial y}\,\mathrm{d}y\,\mathrm{d}x\,.$$

We see that the total flux out of the rectangle is

$$\begin{aligned}
&\int_c^d \langle \boldsymbol{v}(a,y), \boldsymbol{n}(a,y)\rangle \,\mathrm{d}y + \int_a^b \langle \boldsymbol{v}(x,d), \boldsymbol{n}(x,d)\rangle \,\mathrm{d}x \\
&\quad + \int_c^d \langle \boldsymbol{v}(b,y), \boldsymbol{n}(b,y)\rangle \,\mathrm{d}y + \int_a^b \langle \boldsymbol{v}(x,c), \boldsymbol{n}(x,c)\rangle \,\mathrm{d}x \\
&\quad = \int_c^d \int_a^b \frac{\partial v_1}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + \int_c^d \int_a^b \frac{\partial v_2}{\partial y}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_c^d \int_a^b \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_c^d \int_a^b \operatorname{div} \boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}y\,. \qquad (3.33)
\end{aligned}$$

Now consider the polygon $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_8)$ with axis parallel edges to the left in Figure 3.11. The total flux out of the domain inside the polygon is

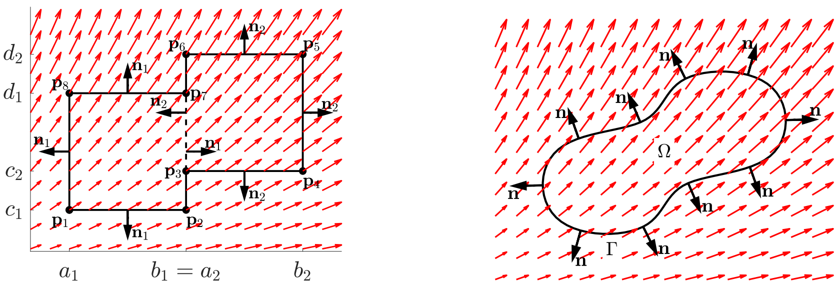

Figure 3.11: Left: The flux out of an axis parallel polygon. Right: The flux out of a domain Ω with boundary Γ.

$$\begin{aligned}
&\int_{a_1}^{b_1} \langle \boldsymbol{n}_1(x,c_1), \boldsymbol{v}(x,c_1)\rangle \,\mathrm{d}x + \int_{c_1}^{c_2} \langle \boldsymbol{n}_1(b_1,y), \boldsymbol{v}(b_1,y)\rangle \,\mathrm{d}y \\
&\quad + \int_{a_2}^{b_2} \langle \boldsymbol{n}_2(x,c_2), \boldsymbol{v}(x,c_2)\rangle \,\mathrm{d}x + \int_{c_2}^{d_2} \langle \boldsymbol{n}_2(b_2,y), \boldsymbol{v}(b_2,y)\rangle \,\mathrm{d}y \\
&\quad + \int_{a_2}^{b_2} \langle \boldsymbol{n}_2(x,d_2), \boldsymbol{v}(x,d_2)\rangle \,\mathrm{d}y + \int_{d_1}^{d_2} \langle \boldsymbol{n}_2(a_2,y), \boldsymbol{v}(a_2,y)\rangle \,\mathrm{d}y \\
&\quad + \int_{a_1}^{b_1} \langle \boldsymbol{n}_1(x,d_1), \boldsymbol{v}(x,d_1)\rangle \,\mathrm{d}y + \int_{c_1}^{d_1} \langle \boldsymbol{n}_1(a_1,y), \boldsymbol{v}(a_1,y)\rangle \,\mathrm{d}y \qquad (3.34)
\end{aligned}$$

If we look at the (dashed) line between the points $\boldsymbol{p}_3$ and $\boldsymbol{p}_7$ then we see that the two normals are in opposite direction and hence $\boldsymbol{n}_1 + \boldsymbol{n}_2 = \boldsymbol{0}$ on that line. As we furthermore have that $b_1 = a_2$ we obtain

$$\begin{aligned}\int_{c_2}^{d_1} \langle \boldsymbol{n}_1(b_1, y), \boldsymbol{v}(b_1, y)\rangle \,\mathrm{d}y + \int_{c_2}^{d_1} \langle \boldsymbol{n}_2(a_2, y), \boldsymbol{v}(a_2, y)\rangle \,\mathrm{d}y \\ = \int_{c_2}^{d_1} \langle \boldsymbol{n}_1(a_2, y) + \boldsymbol{n}_2(a_2, y), \boldsymbol{v}(a_2, y)\rangle \,\mathrm{d}y = \int_{c_2}^{d_1} \langle \boldsymbol{0}, \boldsymbol{v}(a_2, y)\rangle \,\mathrm{d}y = 0\,.\end{aligned}$$

Adding this to (3.34), rearranging, and using (3.33) we can write the total flux as

$$\begin{aligned}\int_{a_1}^{b_1} \langle \boldsymbol{n}_1(x, c_1), \boldsymbol{v}(x, c_1)\rangle \,\mathrm{d}x + \int_{c_1}^{d_1} \langle \boldsymbol{n}_1(b_1, y), \boldsymbol{v}(b_1, y)\rangle \,\mathrm{d}y \\ + \int_{a_1}^{b_1} \langle \boldsymbol{n}_1(x, d_1), \boldsymbol{v}(x, d_1)\rangle \,\mathrm{d}y + \int_{c_1}^{d_1} \langle \boldsymbol{n}_1(a_1, y), \boldsymbol{v}(a_1, y)\rangle \,\mathrm{d}y \\ + \int_{a_2}^{b_2} \langle \boldsymbol{n}_2(x, c_2), \boldsymbol{v}(x, c_2)\rangle \,\mathrm{d}x + \int_{c_2}^{d_2} \langle \boldsymbol{n}_2(b_2, y), \boldsymbol{v}(b_2, y)\rangle \,\mathrm{d}y \\ + \int_{a_2}^{b_2} \langle \boldsymbol{n}_2(x, d_2), \boldsymbol{v}(x, d_2)\rangle \,\mathrm{d}y + \int_{c_1}^{d_2} \langle \boldsymbol{n}_2(a_2, y), \boldsymbol{v}(a_2, y)\rangle \,\mathrm{d}y \\ = \int_{c_1}^{d_1} \int_{a_1}^{b_1} \operatorname{div} \boldsymbol{v} \,\mathrm{d}x \,\mathrm{d}y + \int_{c_2}^{d_2} \int_{a_2}^{b_2} \operatorname{div} \boldsymbol{v} \,\mathrm{d}x \,\mathrm{d}y\,. \quad (3.35)\end{aligned}$$

If we have a general domain $\Omega \subseteq \mathbb{R}^2$ with a piecewise C^1 boundary Γ as to the right in Figure 3.11, the total flux out of the domain is $\int_\Gamma \langle \boldsymbol{v}, \boldsymbol{n}\rangle \,\mathrm{d}s$ where we integrate with respect to arc-length on the boundary curve Γ. By approximating the domain by the union of rectangles that only intersects at the edges, as in the example to the left in Figure 3.11, we see that we have

$$\int_\Gamma \langle \boldsymbol{v}, \boldsymbol{n}\rangle \,\mathrm{d}s = \int_\Omega \operatorname{div} \boldsymbol{v} \,\mathrm{d}x \,\mathrm{d}y\,. \quad (3.36)$$

Arbitrary dimension

If we have a C^1 vector field $\boldsymbol{v}$ on an open set $U \subseteq \mathbb{R}^n$ and an n-dimensional box $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq U$ then a calculation as in (3.33) shows that total flux out of the 2^n sides is

$$\begin{aligned}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\langle \boldsymbol{v}(a_1,x_2,\dots,x_n),\boldsymbol{n}(a_1,x_2,\dots,x_n)\,\mathrm{d}x_2\dots\mathrm{d}x_n\\ +\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\langle \boldsymbol{v}(b_1,x_2,\dots,x_n),\boldsymbol{n}(b_1,x_2,\dots,x_n)\,\mathrm{d}x_2\dots\mathrm{d}x_n\\ \vdots\\ +\int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}}\langle \boldsymbol{v}(x_1,\dots,x_{n-1},b_n),\boldsymbol{n}(x_1,\dots,x_{n-1},b_n)\,\mathrm{d}x_1\dots\mathrm{d}x_{n-1}\\ +\int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}}\langle \boldsymbol{v}(x_1,\dots,x_{n-1},b_n),\boldsymbol{n}(x_1,\dots,x_{n-1},b_n)\,\mathrm{d}x_1\dots\mathrm{d}x_{n-1}\\ =\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}\operatorname{div}\boldsymbol{v}(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n\,. \qquad (3.37)\end{aligned}$$

By stacking such boxes we can approximate any bounded domain $\Omega\subset\mathbb{R}^n$ with a piecewise C^1 boundary and we see that the equations (3.32) and (3.36) are special cases of the following theorem

Theorem 3.50 (Divergence (or Gauss) theorem)**.** *If $\boldsymbol{v}$ is a C^1 vector field on an open set $U\subseteq\mathbb{R}^n$, $\Omega\subseteq U$ is a domain with a piecewise C^1 boundary $S=\partial\Omega$, and $\boldsymbol{n}$ is the outward normal on $\partial\Omega$ then*

$$\int_{\partial\Omega}\langle\boldsymbol{v},\boldsymbol{n}\rangle\,\mathrm{d}S=\int_\Omega\operatorname{div}\boldsymbol{v}\,\mathrm{d}U\,. \qquad (3.38)$$

3.6.2 Stokes theorem

Stokes theorem are concerned with vector fields in dimension three, i.e., with a vector fields defined on some open set $U\subseteq\mathbb{R}^3$. We first need the definition of the *curl* of a vector field.

Definition 3.51. Let $\boldsymbol{v}:U\to\mathbb{R}^3$ be a C^1 vector field defined on some open set $U\subseteq\mathbb{R}^3$. If $\boldsymbol{v}(x,y,z)=(v_1(x,y,z),v_2(x,y,z),v_3(x,y,z))$ then the *curl* of $\boldsymbol{v}$ is a new vector field defined by

$$\operatorname{curl}\boldsymbol{v}=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z},\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x},\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right). \qquad (3.39)$$

The notation $\operatorname{curl}\boldsymbol{v}=\nabla\times\boldsymbol{v}$ is often seen and if ∇ is considered as a vector with components $\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right)$ and formally calculate the cross product $\nabla\times\boldsymbol{v}$ then we obtain $\operatorname{curl}\boldsymbol{v}$, c.f. Exercise 3.4.

Example 3.20. Let $\boldsymbol{v}(x,y,z)=(x,y,z)$. Then $\operatorname{curl}\boldsymbol{v}=\boldsymbol{0}$.

Example 3.21. Let $\boldsymbol{v}(x,y,z)=(-y,x,x^2+y^2+z^2)$. Then $\operatorname{curl}\boldsymbol{v}=(2y,-2x,2)$.

Example 3.22. Let $\boldsymbol{\omega} = (0, 0, \omega)$ and define a vector field by $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\omega} \times \boldsymbol{x}$. Then $\boldsymbol{v}(x, y, z) = (-\omega y, \omega x, 0)$ and $\operatorname{curl} \boldsymbol{v} = (0, 0, 2\omega) = 2\boldsymbol{\omega}$.

A rotation around the z-axis with angular velocity ω is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

The velocity field is

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} &= \begin{pmatrix} -\omega\sin(\omega t) & -\omega\cos(\omega t) & 0 \\ \omega\cos(\omega t) & -\omega\sin(\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \\ &= \begin{pmatrix} -\omega(x_0\sin(\omega t) + y_0\cos(\omega t)) \\ \omega(x_0\cos(\omega t) - y_0\sin(\omega t)) \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned}$$

Now let $\boldsymbol{\omega}$ be an arbitrary non zero vector in space, let $\boldsymbol{x}_0$ be an arbitrary point in space, and define a vector field by $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0)$. By choosing a coordinate system such that $\boldsymbol{x}_0$ is the origin and the z-axis is in the direction of $\boldsymbol{\omega}$ we see that $\operatorname{curl} \boldsymbol{v} = 2\boldsymbol{\omega}$. We also have that the vector field $\boldsymbol{v}$ is the velocity field of a rotation around the z-axis with angular velocity ω.

In other words, the curl of the velocity field of a rotation has the same direction as the axis of rotation and the size is twice the angular velocity.

Remark 3.52. If we interpret a vector field $\boldsymbol{v}$ as the velocity field of fluid and consider a small volume around a point $\boldsymbol{x}$ then $\boldsymbol{v}(\boldsymbol{x})$ describes the instantaneous translation of the volume, $\operatorname{div} \boldsymbol{v}(\boldsymbol{x})$ describes the instantaneous expansion or contraction, and $\operatorname{curl} \boldsymbol{v}(\boldsymbol{x})$ describes the instantaneous rotation of the volume (up to a factor of 2).

Given a surface in $U \subseteq \mathbb{R}^3$ and a vector field $\boldsymbol{v}$ on U. Stokes theorem says that the integral of the normal component of $\operatorname{curl} \boldsymbol{v}$ over the surface is the same as the integral of the tangential component of $\boldsymbol{v}$ along the boundary of the surface.

First a couple of definitions.

Definition 3.53. Let $\epsilon > 0$ and let $\boldsymbol{x} :]a-\epsilon, b+\epsilon[\times]c-\epsilon, d+\epsilon[\to \mathbb{R}^3$ be a C^1 function that parameterise a *regular* surface. Then $\boldsymbol{x}([a, b] \times [c, d])$ is a regular surface with a piecewise C^1 boundary. The boundary of the surface is the image of the boundary of the rectangle.

In Figure 3.12 we have shown an example of a regular surface with boundary. Observe that the orientation of the surface (what way does the normal point) and the boundary (what way does the tangent point) are compatible according to the right hand rule.

By a tedious, but straight forward, calculation we can show Stokes theorem for this kind of surfaces.

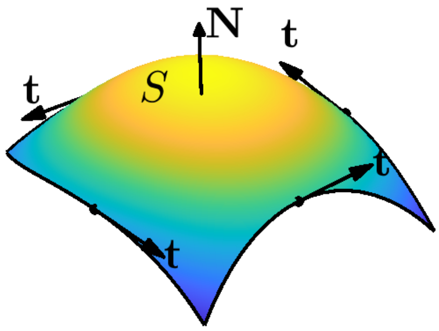

Figure 3.12: A regular surface $S = S_1 \cup S_2$ with boundary.

Lemma 3.54. *Let $U \subseteq \mathbb{R}^3$ be an open set, let $\boldsymbol{v} : U \to \mathbb{R}^3$ be a vector field on U, and let $S \subseteq U$ be a regular surface with a piecewise C^1 boundary and with normal $\boldsymbol{N}$. Then we have*

$$\int_S \langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{N} \rangle \, \mathrm{d}A = \int_{\partial S} \langle \boldsymbol{v}, \boldsymbol{t} \rangle \, \mathrm{d}s \,, \tag{3.40}$$

where $\boldsymbol{t}$ is the tangent of the boundary ∂S.

Proof. Let $\boldsymbol{x} : [a,b] \times [b,c] \to S$ be a parametrisation of S. Then we have

$$\boldsymbol{N} = \frac{\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}}{\left\| \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \right\|} \,, \qquad \mathrm{d}A = \left\| \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \right\| \mathrm{d}u \, \mathrm{d}v \,,$$

and

$$\int_S \langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{N} \rangle \, \mathrm{d}A = \int_a^b \int_c^d \left\langle \operatorname{curl} \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \right\rangle \mathrm{d}u \, \mathrm{d}v \,.$$

We have

$$\begin{aligned}
\left\langle \operatorname{curl} \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v} \right\rangle &= \left\langle \begin{pmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{pmatrix}, \begin{pmatrix} \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \end{pmatrix} \right\rangle \\
&= \frac{\partial v_1}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) - \frac{\partial v_1}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) \\
&\quad + \frac{\partial v_2}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) - \frac{\partial v_2}{\partial z}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) \\
&\qquad + \frac{\partial v_3}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) + \frac{\partial v_3}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) .
\end{aligned}$$

The boundary ∂S consist of four pieces $\boldsymbol{x}(t,c)$, $t \in [a,b]$, $\boldsymbol{x}(b,t)$, $t \in [c,d]$, $\boldsymbol{x}(a+b-t,d)$, $t \in [a,b]$, and $\boldsymbol{x}(a,c+d-t)$, $t \in [c,d]$. The velocity is found by differentiating with respect to t and the tangent $\boldsymbol{t}$ by normalising the velocity. The speed $\frac{\mathrm{d}s}{\mathrm{d}t}$ is the norm of the velocity. In the case of the first piece we have

$$\boldsymbol{t} = \frac{\frac{\partial \boldsymbol{x}(t,c)}{\partial t}}{\left\| \frac{\partial \boldsymbol{x}(t,c)}{\partial t} \right\|} = \frac{\frac{\partial \boldsymbol{x}}{\partial u}(t,c)}{\left\| \frac{\partial \boldsymbol{x}}{\partial u}(t,c) \right\|} \,, \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = \left\| \frac{\partial \boldsymbol{x}}{\partial u}(t,c) \right\| \,,$$

and similar for the other three curves. We obtain

$$\begin{aligned}
\int_{\partial S} \langle \boldsymbol{v}, \boldsymbol{t} \rangle \, \mathrm{d}s &= \int_a^b \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}(t,c)}{\partial t} \right\rangle \mathrm{d}t + \int_c^d \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}(b,t)}{\partial t} \right\rangle \mathrm{d}t \\
&\qquad + \int_a^b \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}(a+b-t,d)}{\partial t} \right\rangle \mathrm{d}t + \int_c^d \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}(a,c+d-t)}{\partial t} \right\rangle \mathrm{d}t \\
&= \int_a^b \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial u}(u,c) \right\rangle \mathrm{d}u + \int_c^d \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial v}(b,v) \right\rangle \mathrm{d}v \\
&\qquad - \int_a^b \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial u}(u,d) \right\rangle \mathrm{d}u - \int_c^d \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial v}(a,v) \right\rangle \mathrm{d}v \\
&= \int_a^b \left(\left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial u}(u,c) \right\rangle - \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial u}(u,d) \right\rangle \right) \mathrm{d}u \\
&\qquad + \int_c^d \left(\left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial v}(b,v) \right\rangle - \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial v}(a,v) \right\rangle \right) \mathrm{d}v \\
&= -\int_a^b \int_c^d \frac{\partial}{\partial v} \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial u}(u,v) \right\rangle \mathrm{d}v \, \mathrm{d}u + \int_c^d \int_a^b \frac{\partial}{\partial u} \left\langle \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial v}(u,v) \right\rangle \mathrm{d}u \, \mathrm{d}v \\
&= -\int_a^b \int_c^d \left(\left\langle \frac{\partial \boldsymbol{v}}{\partial v}, \frac{\partial \boldsymbol{x}}{\partial u} \right\rangle + \left\langle \boldsymbol{v}, \frac{\partial^2 \boldsymbol{x}}{\partial v \partial u} \right\rangle \right) \mathrm{d}v \, \mathrm{d}u \\
&\qquad + \int_a^b \int_c^d \left(\left\langle \frac{\partial \boldsymbol{v}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v} \right\rangle + \left\langle \boldsymbol{v}, \frac{\partial^2 \boldsymbol{x}}{\partial u \partial v} \right\rangle \right) \mathrm{d}v \, \mathrm{d}u \\
&= \int_a^b \int_c^d \left(\left\langle \frac{\partial \boldsymbol{v}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v} \right\rangle - \left\langle \frac{\partial \boldsymbol{v}}{\partial v}, \frac{\partial \boldsymbol{x}}{\partial u} \right\rangle \right) \mathrm{d}v \, \mathrm{d}u \,.
\end{aligned}$$

Now

$$\begin{aligned}
&\left\langle \frac{\partial \boldsymbol{v}}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v} \right\rangle - \left\langle \frac{\partial \boldsymbol{v}}{\partial v}, \frac{\partial \boldsymbol{x}}{\partial u} \right\rangle \\
&\quad = \left\langle \frac{\partial \boldsymbol{v}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \boldsymbol{v}}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \boldsymbol{v}}{\partial z}\frac{\partial z}{\partial u}, \frac{\partial \boldsymbol{x}}{\partial v} \right\rangle - \left\langle \frac{\partial \boldsymbol{v}}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \boldsymbol{v}}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \boldsymbol{v}}{\partial z}\frac{\partial z}{\partial v}, \frac{\partial \boldsymbol{x}}{\partial u} \right\rangle \\
&\quad = \left(\frac{\partial v_1}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial v_1}{\partial z}\frac{\partial z}{\partial u} \right) \frac{\partial x}{\partial v} - \left(\frac{\partial v_1}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial v_1}{\partial z}\frac{\partial z}{\partial v} \right) \frac{\partial x}{\partial u} \\
&\quad + \left(\frac{\partial v_2}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v_2}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial v_2}{\partial z}\frac{\partial z}{\partial u} \right) \frac{\partial y}{\partial v} - \left(\frac{\partial v_2}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v_2}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial v_2}{\partial z}\frac{\partial z}{\partial v} \right) \frac{\partial y}{\partial u} \\
&\quad + \left(\frac{\partial v_3}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v_3}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial v_3}{\partial z}\frac{\partial z}{\partial u} \right) \frac{\partial z}{\partial v} - \left(\frac{\partial v_3}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v_3}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial v_3}{\partial z}\frac{\partial z}{\partial v} \right) \frac{\partial z}{\partial u} \\
&\qquad = \left(\frac{\partial v_1}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial v_1}{\partial z}\frac{\partial z}{\partial u} \right) \frac{\partial x}{\partial v} - \left(\frac{\partial v_1}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial v_1}{\partial z}\frac{\partial z}{\partial v} \right) \frac{\partial x}{\partial u} \\
&\qquad + \left(\frac{\partial v_2}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v_2}{\partial z}\frac{\partial z}{\partial u} \right) \frac{\partial y}{\partial v} - \left(\frac{\partial v_2}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v_2}{\partial z}\frac{\partial z}{\partial v} \right) \frac{\partial y}{\partial u} \\
&\qquad + \left(\frac{\partial v_3}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v_3}{\partial y}\frac{\partial y}{\partial u} \right) \frac{\partial z}{\partial v} - \left(\frac{\partial v_3}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v_3}{\partial y}\frac{\partial y}{\partial v} \right) \frac{\partial z}{\partial u}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial v_1}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) - \frac{\partial v_1}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\\
&+ \frac{\partial v_2}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right) - \frac{\partial v_2}{\partial z}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\\
&+ \frac{\partial v_3}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) + \frac{\partial v_3}{\partial x}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\\
&\qquad\qquad\qquad\qquad = \left\langle \operatorname{curl} \boldsymbol{v}, \frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right\rangle .
\end{aligned}
$$

This finishes the proof. □

Definition 3.55. Suppose we have regular surfaces $S_1, \ldots, S_n$ with piecewise C^1 boundaries $\partial S_1, \ldots, \partial S_n$. If they only intersect at the boundary, i.e., $S_k \cap S_\ell = \partial S_k \cap \partial S_\ell$ if $k \neq \ell$, the union $S = \bigcup_{k=1}^n S_k$ is a piecewise regular surface with a piecewise C^1 boundary.

In Figure 3.13 we have shown an example of a piecewise regular surface $S =$

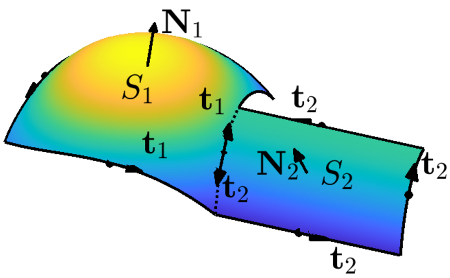

Figure 3.13: A piecewise regular surface $S = S_1 \cup S_2$ with boundary.

$S_1 \cup S_2$ with a piecewise C^1 boundary. The intersection $S_1 \cap S_2$ is the dotted line $\partial S_1 \cap \partial S_2$. Observe the behaviour of the tangent vectors on this interior boundary.

With these preparations it is hard to show Stokes theorem.

Theorem 3.56 (Stokes theorem)**.** *Let $U \subseteq \mathbb{R}^3$ be an open set, let $\boldsymbol{v} : U \to \mathbb{R}^3$ be a vector field on U, and let $S = \bigcup_{k=1}^n S_k \subseteq U$ be a piecewise regular surface with a piecewise C^1 boundary and with normal $\boldsymbol{N}$. Then we have*

$$\int_S \langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{N}\rangle \, \mathrm{d}A = \int_{\partial S} \langle \boldsymbol{v}, \boldsymbol{t}\rangle \, \mathrm{d}s \,, \tag{3.41}$$

where $\boldsymbol{t}$ is the tangent of the boundary ∂S.

Proof. By Lemma 3.54 the theorem holds for each surface S_k, i.e.,

$$\int_S \langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{N}\rangle \, \mathrm{d}A = \sum_{k=1}^n \int_{S_k} \langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{N}\rangle \, \mathrm{d}A = \sum_{k=1}^n \int_{\partial S_k} \langle \boldsymbol{v}, \boldsymbol{t}\rangle \, \mathrm{d}s \,.$$

Now we only have to note that if the normals of the surfaces are in agreement with each other then the tangent vectors of two neighbouring surfaces are opposite of each other. That means that the integrals over the interior boundaries (the dotted curve in Figure 3.13) cancels and we are left with the integral over the exterior boundary (the solid curve in Figure 3.13). □

As integration over the empty set is yields zero we immediately obtain the following result.

Corollary 3.57. *Let $U \subseteq \mathbb{R}^3$ be an open set, let $\boldsymbol{v} : U \to \mathbb{R}^3$ be a vector field on U, and let $S = \bigcup_{k=1}^{n} S_k \subseteq U$ be a piecewise regular surface without boundary, i.e., we have only interior boundaries, and with normal $\boldsymbol{N}$. Then we have*

$$\int_S \langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{N} \rangle \, \mathrm{d}A = 0 \,. \tag{3.42}$$

A surface without a boundary is called a *closed surface*.

Example 3.23. In Example 3.21 we saw that the curl of vector field $\boldsymbol{v}(x, y, z) = (-y, x, x^2 + y^2 + z^2)$ is $\operatorname{curl} \boldsymbol{v} = (2y, -2x, 2)$. Now consider the truncated unit sphere in Figure 3.14. It is cut at $z = \frac{\sqrt{2}}{2}$ so the boundary is a circle with radius

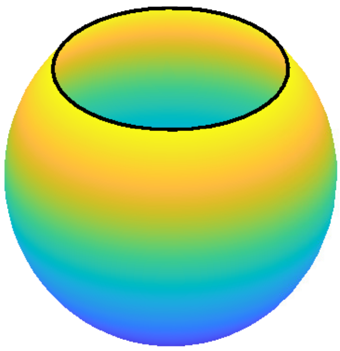

Figure 3.14: The unit sphere cut at $z_0 = \frac{\sqrt{2}}{2}$.

$r_0 = \sqrt{1 - z_0^2} = \frac{\sqrt{2}}{2}$. Suppose we want to find the integral of the normal component of $\operatorname{curl} \boldsymbol{v}$ over this surface. Using Stokes theorem we have

$$\int_S \langle \operatorname{curl} \boldsymbol{v}, \boldsymbol{N} \rangle \, \mathrm{d}A = \int_{\partial S} \langle \boldsymbol{v}, \boldsymbol{t} \rangle \, \mathrm{d}s \,.$$

We can parameterise the boundary ∂S by $\boldsymbol{x}(t) = r_0(\cos t, \sin t, 1)$, $t \in [0, 2\pi]$ (we are using the outward normal). The derivative is $\boldsymbol{x}'(t) = r_0(-\sin t, \cos t, 0)$. Using

(3.24) the line integral of $\boldsymbol{v}$ along the ∂S is

$$\begin{aligned}\int_{\partial S}\langle \boldsymbol{v},\boldsymbol{t}\rangle\,\mathrm{d}s &= \int_0^{2\pi}\langle \boldsymbol{v}(\boldsymbol{x}(t)),\boldsymbol{x}'(t)\rangle\,\mathrm{d}t\\ &= \int_0^{2\pi}\left\langle \begin{pmatrix}-r_0\sin t\\ r_0\cos t\\ 1\end{pmatrix},\begin{pmatrix}-r_0\sin t\\ r_0\cos t\\ 0\end{pmatrix}\right\rangle\,\mathrm{d}t\\ &= \int_0^{2\pi}(r_0^2\sin^2 t + r_0^2\cos^2 t)\,\mathrm{d}t = \int_0^{2\pi} r_0^2\,\mathrm{d}t = 2\pi r_0^2 = 2\pi\frac{2}{4} = \pi\,.\end{aligned}$$

We can get the result even easier. The boundary ∂S is also the boundary of the disc $D = \{(x,y,z) \mid x^2+y^2 \le r_0^2 \wedge z = r_0\}$. The normal of this disc is $\boldsymbol{N}_D = (0,0,1)$ so we have $\langle \operatorname{curl}\boldsymbol{v},\boldsymbol{N}_D\rangle = 2$

$$\begin{aligned}\int_S\langle \operatorname{curl}\boldsymbol{v},\boldsymbol{N}\rangle\,\mathrm{d}A = \int_{\partial S}\langle \boldsymbol{v},\boldsymbol{t}\rangle\,\mathrm{d}s\,. = \int_D\langle \operatorname{curl}\boldsymbol{v},\boldsymbol{N}_D\rangle\,\mathrm{d}A = \int_D 2\,\mathrm{d}A\\ = 2\times\text{area of } D = 2\pi r_0^2 = \pi\,.\end{aligned}$$

3.7 Exercises

Exercise 3.1. Formally calculate $\begin{pmatrix}\frac{\partial}{\partial x_1}\\ \vdots\\ \frac{\partial}{\partial x_n}\end{pmatrix}\cdot\begin{pmatrix}v_1(x_1,\dots,x_n)\\ \vdots\\ v_n(x_1,\dots,x_n)\end{pmatrix}$.

Exercise 3.2. Show that the line integral of a vector field (3.24) does not depend on the parametrisation.

Exercise 3.3. Let $f : [a_1,b_1]\times\cdots\times[a_n,b_n]\to\mathbb{R}$ be uniformly continuous and define $g : [a_1,b_1]\times\cdots\times[a_{n-1},b_{n-1}]\to\mathbb{R}$ by

$$g(x_1,\dots,x_{n-1}) = \int_{a_n}^{b_n} f(x_1,\dots,x_{n-1},t)\,\mathrm{d}t\,.$$

Show that g is uniformly continuous.

Exercise 3.4. Formally calculate $\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}\times\begin{pmatrix}v_1(x,y,z)\\ v_2(x,y,z)\\ v_3(x,y,z)\end{pmatrix}$.

Exercise 3.5. Check Example 3.20 and 3.21.

Exercise 3.6. What is the result in Example 3.23 if we cut the unit sphere at $z = h$ for some $h \in [-1,1]$?

Bibliography

[1] Beelen, Peter, *01001 Mathematics 1a*, Lecture notes, DTU Compute, 2023, https://01001.compute.dtu.dk/_assets/enotesvol1.pdf,

Appendix A

More on the real numbers

A.1 Ordered fields

What kinds of numbers do we know?

- The *natural numbers* or *positive integers*:

$$\mathbb{N} = \mathbb{Z}_+ = \{1, 2, 3, \dots\}. \tag{A.1}$$

- The *negative integers*:

$$\mathbb{Z}_- = \{-1, -2, -3, \dots\}. \tag{A.2}$$

- The *non negative integers*:

$$\mathbb{Z}_0 = \{0, 1, 2, 3, \dots\}. \tag{A.3}$$

- The *integers*:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}. \tag{A.4}$$

- The *rational numbers*:

$$\mathbb{Q} = \left\{ \frac{p}{q} \middle| p \in \mathbb{Z}, q \in \mathbb{N} \right\}. \tag{A.5}$$

- The real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$.

We always consider these different sets of numbers as subsets of each other: $\mathbb{N} \subset \mathbb{Z}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. In [1] we saw how to get from the real numbers $\mathbb{R}$ to the complex numbers $\mathbb{C}$. And on the intuitive level it is perhaps obvious how first to get from the natural numbers $\mathbb{N}$ to $\mathbb{Z}_0$, from $\mathbb{Z}_0$ to the integers $\mathbb{Z}$ and then to the rational numbers $\mathbb{Q}$. There are mathematical precise algebraic constructions for each of these steps, but we will not describe those.

The step from $\mathbb{Q}$ to $\mathbb{R}$ is different in nature. On the intuitive level we discover that we miss some numbers like $\sqrt{2}$ and π so we include those, but exactly what numbers are "missing" and how do we add and multiply with them? An other, geometrical or physical, viewpoint is to consider the rational numbers as certain points on a line and then consider the real numbers as all points on the line, but again, it is not obvious how to add and multiply points on a line.

The six axioms of an ordered field (Definition 1.1) implies other well known properties. Using (1.5) (with $c = -a$) we have

$$\forall a \in \mathbb{F} : 0 \leq a \implies -a \leq 0 \,, \qquad \forall a \in \mathbb{F} : a \leq 0 \implies 0 \leq -a \,. \tag{A.6}$$

As the ordering is total this implies

$$\forall a \in \mathbb{F} : 0 \leq a \vee 0 \leq -a \,. \tag{A.7}$$

As $0 \cdot b = 0$ for all b and using (1.6) we have

$$\forall a, b \in \mathbb{F} : 0 \leq a \wedge 0 \leq b \implies 0 \leq a \cdot b \,. \tag{A.8}$$

As $a^2 = (-a)^2$ this and (A.7) implies

$$\forall a \in \mathbb{F} : 0 \leq a^2 \,. \tag{A.9}$$

As $1^2 = 1$ this in turn implies

$$0 \leq 1 \,. \tag{A.10}$$

If $0 \leq a$, $a \neq 0$ and $a^{-1} \leq 0$ then we have $1 = a \cdot a^{-1} \leq a \cdot 0 = 0$ but that would imply that $1 = 0$. So we must have

$$\forall a \in \mathbb{F} \setminus \{0\} : 0 \leq a \implies 0 \leq a^{-1} \,. \tag{A.11}$$

Given the ordering "$\leq$" on the field $\mathbb{F}$ we can define new relations "$<$", "$\geq$", and "$>$" on $\mathbb{F}$ by

$$a < b \iff a \leq b \wedge a \neq b \,, \quad a \geq b \iff b \leq a \,, \quad a > b \iff b < a \,. \tag{A.12}$$

They satisfied the expected rules know from $\mathbb{Q}$ and $\mathbb{R}$:

$$\forall a \in \mathbb{F} : a \geq a \,, \tag{A.13}$$

$$\forall a, b \in \mathbb{F} : a \geq b \wedge b \geq a \implies a = b \,, \tag{A.14}$$

$$\forall a, b, c \in \mathbb{F} : a \geq b \wedge b \geq c \implies a \geq c \,, \tag{A.15}$$

$$\forall a, b \in \mathbb{F} : a \geq b \vee b \geq a \,, \tag{A.16}$$

$$\forall a, b, c \in \mathbb{F} : a \geq b \implies a + c \geq b + c \,, \tag{A.17}$$

$$\forall a, b, c \in \mathbb{F} : a \geq b \wedge c \geq 0 \implies a \cdot c \geq b \cdot c \,. \tag{A.18}$$

As $a \neq b \Rightarrow a + c \neq b = c$ for all $a, b, c \in \mathbb{F}$ we have

$$\forall a, b, c \in \mathbb{F} : a < b \wedge b < c \implies a < c\,, \tag{A.19}$$
$$\forall a, b, c \in \mathbb{F} : a < b \implies a + c < b + c\,, \tag{A.20}$$
$$\forall a, b, c \in \mathbb{F} : a < b \wedge 0 < c \implies a \cdot c < b \cdot c\,. \tag{A.21}$$
$$\forall a, b, c \in \mathbb{F} : a > b \wedge b > c \implies a > c\,, \tag{A.22}$$
$$\forall a, b, c \in \mathbb{F} : a > b \implies a + c > b + c\,, \tag{A.23}$$
$$\forall a, b, c \in \mathbb{F} : a > b \wedge c > 0 \implies a \cdot c > b \cdot c\,. \tag{A.24}$$

Every total ordered field contains a copy of the integers.

Lemma A.1. *If $(\mathbb{F}, +, \cdot, \leq)$ is an ordered field then we have a unique map $f : \mathbb{Z} \to \mathbb{F}$ with $f(1) \neq 1$ that preserves the addition, the multiplication, and the ordering. That is, for all $n, m \in \mathbb{Z}$ we have $f(n + m) = f(n) + f(m)$, $f(nm) = f(n) \cdot f(m)$, and $n \leq m \Rightarrow f(n) \leq f(m)$.*

Proof. To prove existence we define f on $\mathbb{Z}$ recursively by letting

$$\begin{aligned} &f(0) = 0\,, && \\ &f(n) = f(n-1) + 1\,, && \text{for } n \in \mathbb{Z}_+\,, \\ &f(n) = f(n+1) - 1\,, && \text{for } n \in \mathbb{Z}_-\,. \end{aligned}$$

Observe that this definition immediately tells us that $f(1) = 1$, $f(-1) = -1$, and more general that $f(n+1) = f(n) + 1$ and $f(n-1) = f(n) - 1$ for all $n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}_0$ we want to prove that $f(-n) = -f(n)$ using induction on n. The case $n = 0$ is trivial and if $f(-n) = -f(n)$ for some $n \in \mathbb{Z}_0$ then

$$\begin{aligned} f(-(n+1)) = f(-n-1) = f(-n) - 1 = -f(n) - f(1) \\ = -(f(n) + f(1)) = -f(n+1)\,. \end{aligned}$$

If $n \in \mathbb{Z}_-$ then $-n \in \mathbb{Z}_+$ and we have $-f(n) = -f(-(-n)) = -(-f(-n)) = f(-n)$, i.e., the equation $f(-n) = -f(n)$ holds for all $n \in \mathbb{Z}$.

Let $n, m \in \mathbb{Z}$. We first want to prove that $f(n + m) = f(n) + f(m)$ using induction on m. The case $m = 0$ is trivial so assume it is true for some $m \in \mathbb{Z}_0$ then

$$f(n + m + 1) = f(n + m) + 1 = f(n) + f(m) + f(1) = f(n) + f(m + 1)\,.$$

Similar if it is true for some $m \in \mathbb{Z}_-$ then

$$f(n + m - 1) = f(n + m) - 1 = f(n) + f(m) + f(-1) = f(n) + f(m - 1)\,,$$

and we are done. Next we want to prove that $f(m \cdot n) = f(m) \cdot f(n)$ for all $n, m \in \mathbb{Z}$. Again by induction on m. The case $m = 0$ is trivial and if it is true for some $m \in \mathbb{Z}_0$ then

$$\begin{aligned} f((m+1) \cdot n) = f(m \cdot n + n) = f(m \cdot n) + f(n) \\ = f(m) \cdot f(n) + f(1) \cdot f(n) = (f(m) + f(1)) \cdot f(n) = f(m+1) \cdot f(n)\,. \end{aligned}$$

Similar, if it is true for some $m \in \mathbb{Z}_-$ then

$$\begin{aligned} f((m-1)\cdot n) &= f(m\cdot n - n) = f(m\cdot n) - f(n) \\ &= f(m)\cdot f(n) - f(1)\cdot f(n) = (f(m)-f(1))\cdot f(n) = f(m-1)\cdot f(n)\,, \end{aligned}$$

and we are done. At this point we have a map $\mathbb{Z} \to \mathbb{F}$ that preserves addition and multiplication. But we still need to prove that the ordering is preserved.

First we show that if $0 \le n$ then $0 \le f(n)$. We do it by induction on n. The case $n = 0$ is trivial, so assume it is true for an $n \in \mathbb{Z}_0$, i.e., that $0 \le f(n)$. Then we have $0 \le 1 \le f(n) + 1 = f(n+1)$ and we are done. If we now have $n, m \in \mathbb{Z}$ with $n \le m$ then $0 \le m - n$ and hence $0 \le f(m-n) = f(m) - f(n)$. Adding $f(n)$ yields $f(n) \le f(m)$ as required.

Uniqueness follows from the fact that the condition $f(n) = f(n+0) = f(n)+f(0)$ implies that $f(0) = 0$. Likewise, the condition $f(1) = f(1\cdot 1) = f(1)\cdot f(1)$ implies that $f(1) = 0$ or $f(1) = 1$. As $f(1) \ne 0$ we must have $f(1) = 1$. The conditions $f(n \pm 1) = f(n) \pm f(1) = f(n) \pm 1$ implies that our recursive definition is the only possibility. □

Remark A.2. It is not hard to see that f is injective, c.f. Exercise A.1.

Every total ordered field contains the rational numbers as a subfield.

Theorem A.3. *If* $(\mathbb{F}, +, \cdot, \le)$ *is an ordered field then we have a unique map* $f : \mathbb{Q} \to \mathbb{F}$*, with* $f(1) \ne 0$ *that preserves the addition, the multiplication, and the ordering. That is, for all* $x, y \in \mathbb{Q}$ *we have* $f(x+y) = f(x) + f(y)$*,* $f(x\cdot y) = f(x)\cdot f(y)$*, and* $x \le y \Rightarrow f(x) \le f(y)$*.*

Proof. Lemma A.1 shows that we have a unique map $f : \mathbb{Z} \to \mathbb{F}$ with the required properties. If we can extend this map to $\mathbb{Q}$ then we must have $f\left(\frac{p}{q}\right) = f(pq^{-1}) = f(p)\cdot f(q^{-1}) = f(p)\cdot f(q)^{-1}$. So the only possibility is to define f on $\mathbb{Q}$ by letting $f\left(\frac{p}{q}\right) = f(p)\cdot f(q)^{-1}$ for $(p, q) \in \mathbb{Z} \times \mathbb{N}$. Observe that if $n \in \mathbb{N}$ then

$$\begin{aligned} f\left(\frac{np}{nq}\right) &= f(np)\cdot f(nq)^{-1} = f(n)\cdot f(p)\cdot (f(n)\cdot f(q))^{-1} \\ &= f(n)\cdot f(p)\cdot f(n)^{-1}\cdot f(q)^{-1} = f(p)\cdot f(q)^{-1} = f\left(\frac{p}{q}\right)\,, \end{aligned}$$

so the map is well defined. We now have

$$\begin{aligned} f\left(\frac{p}{q}+\frac{m}{n}\right) &= f\left(\frac{pn+qm}{qn}\right) = f(pn+qm)f(qn)^{-1} \\ &= (f(p)\cdot f(n) + f(q)\cdot f(m))\cdot (f(q)\cdot f(n))^{-1} \\ &= f(p)\cdot f(n)\cdot f(q)^{-1}\cdot f(n)^{-1} + f(q)\cdot f(m)\cdot f(q)^{-1}\cdot f(n)^{-1} \\ &= f(p)\cdot f(q)^{-1} + f(m)\cdot f(n)^{-1} = f\left(\frac{p}{q}\right) + f\left(\frac{m}{n}\right)\,, \end{aligned}$$

and

$$f\left(\frac{p}{q}\frac{m}{n}\right) = f\left(\frac{pm}{qn}\right) = f(pm)\cdot f(qn)^{-1} = f(p)\cdot f(m)\cdot f(q)^{-1}\cdot f(n)^{-1}$$
$$= f(p)\cdot f(q)^{-1}\cdot f(m)\cdot f(n)^{-1} = f\left(\frac{p}{q}\right)\cdot f\left(\frac{m}{n}\right).$$

So we now have a well defined map $\mathbb{Q}\to\mathbb{F}$ that preserves addition and multiplication.

If $q\in\mathbb{N}$ then $f(q)\neq 0$, $0\leq f(q)$ and $0\leq f(q)^{-1}$. So if $0\leq\frac{p}{q}$ then $0\leq p$. Hence $0\leq f(p)$ and $0=0\cdot f(q)^{-1}\leq f(p)\cdot f(q)^{-1}=f\left(\frac{p}{q}\right)$.

If $\frac{m}{n}\leq\frac{p}{q}$ then $n,q>0$ and $qm\leq np$. Now $f(n),f(q)>0$ and $f(q)f(m)=f(qm)\leq f(np)=f(n)f(p)$. Multiplying with the positive number $f(n)^{-1}f(q)^{-1}$ yields $f\left(\frac{m}{n}\right)=f(n)^{-1}f(m)\leq f(q)^{-1}f(p)=f\left(\frac{p}{q}\right)$. □

A.2 Infimum and supremum

Some subsets of $\mathbb{Q}$ or $\mathbb{R}$ have a smallest and/or a largest element some do not. The closed interval $[0,1]$ has both a minimum (0) and a maximum (1). The half open interval $[0,1[$ has a minimum (0), but no maximum and the open interval $]0,1[$ has neither a minimum nor a maximum.

Definition A.4. Let $(\mathbb{F},\leq)$ be an ordered set and let $A\subset\mathbb{F}$. If there exist an element $a\in A$ such that $a\leq x$ for all $x\in A$ then we say a is the *minimum* of A and we write $a=\min A$. If there exist an element $a\in A$ such that $x\leq a$ for all $x\in A$ then we say a is the *maximum* of A and we write $a=\max A$.

Definition A.5. Let $(\mathbb{F},\leq)$ be an ordered set and let $A\subset\mathbb{F}$. If there exist an element $a\in\mathbb{F}$ such that $a\leq x$ for all $x\in A$ then we say A is *bounded from below* and we say a is a *lower bound* for A. If there exist an element $a\in\mathbb{F}$ such that $x\leq a$ for all $x\in A$ then we say A is *bounded from above* and we say a is a *upper bound* for A.

The intervals $[0,1]$, $[0,1[$, $]0,1[$ has the infimum 0 and the supremum 1.

Definition A.6. Let $(\mathbb{F},\leq)$ be an ordered set and let $A\subset\mathbb{F}$. If the set of lower bounds for A has a maximum a then we say a is the *infimum* of A and we write $a=\inf A$. If the set of upper bounds for A has a minimum a then we say a is the *supremum* of A and we write $a=\sup A$.

So if it exists then the infimum is the largest lower bound. Similar, if it exists the supremum is the smallest upper bound.

Lemma A.7. *Let $(\mathbb{F},+,\cdot,\leq)$ be an ordered field. If the set $\left\{\frac{1}{n}\middle|n\in\mathbb{N}\right\}\subseteq\mathbb{F}$ has an infimum then it is* 0.

Proof. Assume the opposite, i.e., that $a = \inf\left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\}$ and $a \neq 0$. As 0 is a lower bound we have $0 < a$ and hence $a < a + a = 2 \cdot a$. As a is the largest lower bound $2 \cdot a$ is not a lower bound so we can find $n \in \mathbb{N}$ such that $\frac{1}{n} < 2 \cdot a$. But $\frac{1}{2n} = \frac{1}{2} \cdot \frac{1}{n} < \frac{1}{2} \cdot 2 \cdot a = a$ contradicting that a is a lower bound. □

The lemma implies the following results

Corollary A.8. *Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field. If the set $\left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\} \subseteq \mathbb{F}$ has an infimum and $x, y \in \mathbb{F}$ with $x < y$ then there exist a $n \in \mathbb{N}$ such that $\frac{1}{n} \leq y - x$.*

Proof. We have $0 < y - x$ and 0 is the largest lower bound, so $y - x$ is not a lower bound and the result follows. □

Corollary A.9. *Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field. If the set $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \subseteq \mathbb{F}$ has an infimum and $x \in \mathbb{F}$ then there exist a $n \in \mathbb{N}$ such that $x \leq n$.*

Proof. Assume the opposite, i.e., that we have an $x \in \mathbb{F}$ such that $n \leq x$ for all $n \in \mathbb{N}$. Then $0 < x$ and hence $0 < x^{-1}$ we also have for all $n \in \mathbb{N}$ that $x^{-1} = x^{-1} \cdot \frac{1}{n} \cdot n \leq x^{-1} \cdot \frac{1}{n} \cdot x = \frac{1}{n}$. But that means that x^{-1} is a lower bound, contradicting that the infimum is 0. □

If $(\mathbb{F}, +, \cdot, \leq)$ is an ordered field and $a, b \in \mathbb{F}$ with $a \leq b$ then we can define the intervals

$$
\begin{aligned}
[a, b] &= \{x \in \mathbb{F} \mid a \leq x \wedge x \leq b\}, & [a, b[&= \{x \in \mathbb{F} \mid a \leq x \wedge x < b\}, \\
]a, b] &= \{x \in \mathbb{F} \mid a < x \wedge x \leq b\}, &]a, b[&= \{x \in \mathbb{F} \mid a < x \wedge x < b\},
\end{aligned}
$$

The first is called a closed interval, the last an open intervals and the other two are called half open intervals. We also introduce the following notation for *half lines*:

$$
\begin{aligned}
[a, \infty[&= \{x \in \mathbb{F} \mid a \leq x\}, &]-\infty, b] &= \{x \in \mathbb{F} \mid x \leq b\}, \\
]a, \infty[&= \{x \in \mathbb{F} \mid a < x\}, &]-\infty, b[&= \{x \in \mathbb{F} \mid x < b\},
\end{aligned}
$$

Corollary A.10. *Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field. If the set $\left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\} \subseteq \mathbb{F}$ has an infimum and $x \in \mathbb{F}$ then there exist a $n \in \mathbb{N}$ such that $x \in [n, n+1[$.*

Proof. By Corollary A.9 we can find $n_1, n_2 \in \mathbb{N}$ such that $x \leq n_2$ and $-x \leq n_1$ and hence $x \in [-n_1, n_2]$. As $[-n_1, n_2] \subseteq \bigcup_{n=-n_1}^{n_2} [n, n+1[$ we have an $n \in \{-n_1, -n_1 + 1, \ldots . n_2\}$ such that $x \in [n, n+1[$. □

Theorem A.11. *Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field. The the following three conditions are equivalent*

1. *Every non empty subset $A \subseteq \mathbb{F}$ that is bounded from below has an infimum.*

2. *Every non empty subset $A \subseteq \mathbb{F}$ that is bounded from above has a supremum.*

3. *If $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots \supseteq [a_n, b_n] \supseteq \dots$ is a nested sequence of closed intervals such that $\forall k \in \mathbb{N}\, \exists n_0 \in \mathbb{N} : n > n_0 \Rightarrow b_n - a_n \leq \frac{1}{k}$ then there exists $a \in \mathbb{F}$ such that $\bigcap_{n\in\mathbb{N}}[a_n, b_n] = \{a\}$. (The nested interval theorem).*

Proof. If $A \subseteq \mathbb{F}$ then we put $-A = \{x \in \mathbb{F} \mid -x \in A\}$, i.e., we simply multiply all elements of A by -1. If a is a lower bound for A then $-a$ is an upper bound for $-A$. So if B is the set of lower bounds for A then $-B$ is the set of upper bounds for $-A$. Also if b is a maximum for B then $-b$ is a minimum for $-B$. So if A has an infimum then $-A$ has a supremum and $\sup -A = -\inf A$. This shows that 1. and 2. are equivalent.

2. $\Rightarrow$ 3.: The set $\{a_n \mid n \in \mathbb{N}\}$ is bounded from above by b_1 so we can put $a = \sup\{a_n \mid n \in \mathbb{N}\}$. Now a is an upper bound for $\{a_n \mid n \in \mathbb{N}\}$ (the smallest) so $a_n \leq a$ for all $n \in \mathbb{N}$. Furthermore, every b_n is an upper bound for $\{a_n \mid n \in \mathbb{N}\}$ so $a \leq b_n$ for all $n \in \mathbb{N}$. That is, $a \in [a_n, b_n]$ for all $n \in \mathbb{N}$ and hence $a \in \bigcap_{n\in\mathbb{N}}[a_n, b_n]$.

3. $\Rightarrow$ 1.: Let $A \subseteq \mathbb{F}$ be non empty and bounded from below. Choose a lower bound a_1 for A and an element $b_1 \in A$. If a_1 is the largest lower bound we are done ($\inf A = a_1$). If b_1 is the minimum of A we are done ($\inf A = b_1$). Otherwise consider $\frac{1}{2}(a_1 + b_1) \in \mathbb{F}$. If it is a lower bound for A we put $a_2 = \frac{1}{2}(a_1 + b_1)$ and $b_2 = b_1$. Otherwise we can find $b_2 \in A$ such that $b_2 < \frac{1}{2}(a_1 + b_1)$ and we put $a_2 = a_1$. In the first case we have $b_2 - a_2 = b_1 - \frac{1}{2}(a_1 + b_1) = \frac{b_1 - a_1}{2}$ and in the second case we have $b_2 - a_2 \leq \frac{1}{2}(a_1 + b_1) - a_1 = \frac{b_1 - a_1}{2}$. In both cases $b_2 - a_2 \leq \frac{b_1 - a_1}{2}$.

Continuing this we way we either stop because we have found the infimum or we have a set of lower bounds $\{a_n \in \mathbb{F} \mid n \in \mathbb{N}\}$ for A and a set of elements $\{b_n \in A \mid n \in \mathbb{N}\}$ in A such that $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$, i.e., $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. Furthermore, $b_n - a_n \leq \frac{b_1 - a_1}{2^n}$.

First we note that 0 is a lower bound for $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and any larger lower bound must be an element of $\left[0, \frac{1}{n}\right]$ for all $n \in \mathbb{N}$. As $0 \in \bigcap_{n\in\mathbb{N}}\left[0, \frac{1}{n}\right]$ and hence $\bigcap_{n\in\mathbb{N}}\left[0, \frac{1}{n}\right] = \{0\}$ we see that 0 is the largest lower bound, i.e., $0 = \inf\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Now Corollary A.9 tells us that there exists a $K \in \mathbb{N}$ such that $b_1 - a_1 \leq K$ and hence $b_n - a_n \leq \frac{K}{2^n}$. By Condition 3 there exists an element $a \in \mathbb{F}$ such that $\bigcap_{n\in\mathbb{N}}[a_n, b_n] = \{a\}$. We want to show that a is the infimum for A.

First we show that a is a lower bound. Assume the opposite, then there exists an $x \in A$ such that $x < a$. By Corollary A.8 we have a $k \in \mathbb{N}$ such that $\frac{1}{k} \leq a - x$. Next we can find an $n \in \mathbb{N}$ such that $b_n - a_n \leq \frac{K}{2^n} < \frac{1}{k} \leq a - x$. This implies that $x < a_n + a - b_n$. But $a \in [a_n, b_n]$ so $a \leq b_n$ and hence $x < a_n$ contradicting that a_n is a lower bound for A.

Finally we show that a is the largest lower bound. Assume the opposite, then there exists a lower bound $x \in \mathbb{F}$ such that $a < x$. By Corollary A.8 we have a $k \in \mathbb{N}$ such that $\frac{1}{k} \leq x - a$. Next we can find an $n \in \mathbb{N}$ such that $b_n - a_n \leq \frac{K}{2^n} < \frac{1}{k} \leq x - a$. This implies that $x > b_n + a - a_n$. But $a \in [a_n, b_n]$ so $a_n \leq a$ and hence $x > b_n$ contradicting that x is a lower bound for A. □

Recall that according to Theorem A.3 we can consider the rational numbers as a sub field of any ordered field. We first show that if an ordered field $\mathbb{F}$ has the properties in Theorem A.11 then $\mathbb{Q}$ is dense in $\mathbb{F}$ in the following sense:

Lemma A.12. *Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field where all subsets bounded from below has an infimum. If $x, y \in \mathbb{F}$ and $x < y$ then there exist $\frac{p}{q} \in \mathbb{Q}$ such that $\frac{p}{q} \in [x, y]$.*

Proof. As $x < y$ Corollary A.8 tells us that we can find an $n \in \mathbb{N}$ such that $\frac{1}{n} \leq y - x$ and hence $1 \leq n \cdot y - n \cdot x$. Corollary A.10 give us $n_1, n_2 \in \mathbb{N}$ such that $n \cdot x \in [n_1, n_1 + 1[$ and $n \cdot y \in [n_2, n_2 + 1[$. As $1 \leq n \cdot y - n \cdot x$ we must have $n_1 + 1 \leq n_2$. We now have $nx < n_2 \leq y$ and hence $\frac{n_2}{n} \in [x, y]$. ☐

Corollary A.13. *Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field where all subsets bounded from below has an infimum. If $x \in \mathbb{F}$ then $x = \inf \left\{ \frac{p}{q} \in \mathbb{Q} \middle| x < \frac{p}{q} \right\}$.*

Proof. Let $A = \left\{ \frac{p}{q} \in \mathbb{Q} \middle| x < \frac{p}{q} \right\}$. As x is a lower bound for A we have an infimum $y = \inf A$ and $x \leq y$. If $x \neq y$ then Lemma A.12 tells us that we can find an $\frac{p}{q} \in \mathbb{Q}$ such that $\frac{p}{q} \in [x, \frac{x+y}{2}]$, but then $\frac{p}{q} < y$ contradicting that y is a lower bound. That is, we must have $x = y = \inf A$. ☐

If we accept that $\mathbb{R}$ satisfies the nested interval theorem then this property completely characterise the real numbers.

Theorem A.14. *If $(\mathbb{F}, +, \cdot, \leq)$ is an ordered field where all subsets bounded from below has an infimum. Then there exist a unique isomorphism $\mathbb{R} \to \mathbb{F}$, i.e., a unique bijective map $f : \mathbb{R} \to \mathbb{F}$ such that we for all $x, y \in \mathbb{R}$ have that $f(x+y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$, and $x \leq y \Rightarrow f(x) \leq f(y)$.*

Proof. First uniqueness: Assume we have an isomorphism $\mathbb{R} \to \mathbb{F}$. We have the rational numbers as $\mathbb{Q}$ subfields of both $\mathbb{R}$ and $\mathbb{F}$. If we restrict f to $\mathbb{Q} \subset \mathbb{R}$ then by Theorem A.3 it has to be the identity, i.e., if $\frac{p}{q} \in \mathbb{Q}$ then $f\left(\frac{p}{q}\right) = \frac{p}{q}$. Let $x \in \mathbb{R}$ and consider the set $A = \mathbb{Q} \cap [x, \infty[\subset \mathbb{R}$. As f preserves the ordering a lower bound for a is mapped to a lower bound for $f(A)$, i.e., we have $f(x) \leq \inf f(A) = \inf\left\{ f\left(\frac{p}{q}\right) \in \mathbb{F} \,\middle|\, x \leq \frac{p}{q} \right\}$. As the same is true for the set of lower limits of A we must have

$$f(x) = \inf f(A) = \inf \left\{ \frac{p}{q} \in \mathbb{F} \middle| p \in \mathbb{Z} \wedge q \in \mathbb{N} \wedge x \leq \frac{p}{q} \right\} .$$

We now have uniqueness and we also have a well defined map $f : \mathbb{R} \to \mathbb{F}$. We need to show that f is bijective and preserves addition, multiplication, and the ordering.

The ordering is preserved: Let $x, y \in \mathbb{R}$ and assume $x < y$. Then we can find $\frac{m}{n} \in \mathbb{Q}$ such that $x < \frac{m}{n} < y$. But then

$$\inf \left\{ \frac{p}{q} \in \mathbb{F} \middle| p \in \mathbb{Z} \wedge q \in \mathbb{N} \wedge x \leq \frac{p}{q} \right\} < \frac{m}{n} \\ < \inf \left\{ \frac{p}{q} \in \mathbb{F} \middle| p \in \mathbb{Z} \wedge q \in \mathbb{N} \wedge y \leq \frac{p}{q} \right\},$$

i.e., $f(x) < f(y)$. This also proves injectivity.

Surjective: Let $y \in \mathbb{F}$ and consider $A = \left\{ \frac{p}{q} \in \mathbb{F} \,\middle|\, p \in \mathbb{Z} \wedge q \in \mathbb{N} \wedge y \leq \frac{p}{q} \right\}$. We can consider A as a subset of $\mathbb{Q}$ and hence as a subset of $\mathbb{R}$ if we put $x = \inf\left\{ \frac{p}{q} \in \mathbb{R} \,\middle|\, p \in \mathbb{Z} \wedge q \in \mathbb{N} \wedge y \leq \frac{p}{q} \right\}$. Then we have $f(x) = y$.

Preserves addition: Let $x, y \in \mathbb{R}$. Put $A = \mathbb{Q} \cap [x, \infty[$, $B = \mathbb{Q} \cap [y, \infty[$, and define

$$A + B = \left\{ \frac{p}{q} + \frac{m}{n} \in \mathbb{Q} \middle| \frac{p}{q} \in A \wedge \frac{m}{n} \in B \right\}.$$

If a and b are lower bounds for A and B, respectively then $a + b$ is a lower bound for $A + B$ using this is not hard to see that $x + y = \inf A + \inf B = \inf(A + B)$ and

$$\begin{aligned} f(x+y) = \inf f(A+B) &= \inf \left\{ \frac{p}{q} + \frac{m}{n} \in \mathbb{Q} \middle| \frac{p}{q} \in A \wedge \frac{m}{n} \in B \right\} \\ &= \inf \left\{ \frac{p}{q} + \frac{m}{n} \in \mathbb{Q} \middle| x \leq \frac{p}{q} \wedge y \leq \frac{m}{n} \right\} \\ &= \inf \left\{ \frac{p}{q} \in \mathbb{Q} \middle| x \leq \frac{p}{q} \right\} + \inf \left\{ \frac{m}{n} \in \mathbb{Q} \middle| y \leq \frac{m}{n} \right\} = f(x) + f(y). \end{aligned}$$

Preserves multiplication: First we note that $x + (-x) = 0$ implies that $f(x) + f(-x) = f(0) = 0$, i.e., $f(-x) = -f(x)$.

Let $x, y \in \mathbb{R}$ if either x or y is zero we clearly have $f(xy) = f(0) = 0 = f(x) \cdot f(y)$. So we first assume that $x, y > 0$ and let $A = \mathbb{Q} \cap [x, \infty[$ and $B = \mathbb{Q} \cap [y, \infty[$. We now define

$$A \cdot B = \left\{ \frac{p}{q} \frac{m}{n} \in \mathbb{Q} \middle| \frac{p}{q} \in A \wedge \frac{m}{n} \in B \right\}.$$

If a and b are lower bounds for A and B, respectively then ab is a lower bound for. $A \cdot B$ using this is not hard to see that $xy = \inf A \cdot \inf B = \inf(A \cdot B)$ and

$$\begin{aligned} f(xy) = \inf f(A \cdot B) &= \inf \left\{ \frac{p}{q} \cdot \frac{m}{n} \in \mathbb{Q} \middle| \frac{p}{q} \in A \wedge \frac{m}{n} \in B \right\} \\ &= \inf \left\{ \frac{p}{q} \cdot \frac{m}{n} \in \mathbb{Q} \middle| x \leq \frac{p}{q} \wedge y \leq \frac{m}{n} \right\} \\ &= \inf \left\{ \frac{p}{q} \in \mathbb{Q} \middle| x \leq \frac{p}{q} \right\} \cdot \inf \left\{ \frac{m}{n} \in \mathbb{Q} \middle| y \leq \frac{m}{n} \right\} = f(x) \cdot f(y). \end{aligned}$$

We still miss the cases where one or both numbers are negative. We have

$$f((-x)y) = f(-xy) = -f(xy) = -f(x) \cdot f(y) = f(-x) \cdot f(y).$$

Similar

$$\begin{aligned} f((-x)(-y)) = f(xy) = f(x)f(y) \\ &= (-f(-x)) \cdot (-f(-y)) = f(-x) \cdot f(-y). \quad \square \end{aligned}$$

Before we give a construction of the real numbers we show a perhaps surprising consequence of the nested interval theorem. We have seen that the rational numbers are dense in the real numbers. But in a sense that will be made precise below there are extremely more real numbers that rational numbers. We can count the rational numbers but not the real numbers.

Definition A.15. A set A is called *countable* if it is finite or there exists a bijective map $\mathbb{N} \to A$. Otherwise A is called *uncountable*

Clearly $\mathbb{N}$ is countable. We define a bijective map $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} \frac{n}{2}\,, & \text{if } n \text{ is even}\,, \\ \frac{1-n}{2}\,, & \text{if } n \text{ is odd}\,, \end{cases}$$

so $\mathbb{Z}$ is countable. The sequence $f(1), f(2), \cdots$ is $0, 1, -1, 2, -2, 3, \cdots$. But also $\mathbb{Z} \times \mathbb{Z}$ is countable, see Figure A.1 left, where we visit each point of $\mathbb{Z} \times \mathbb{Z}$ exactly

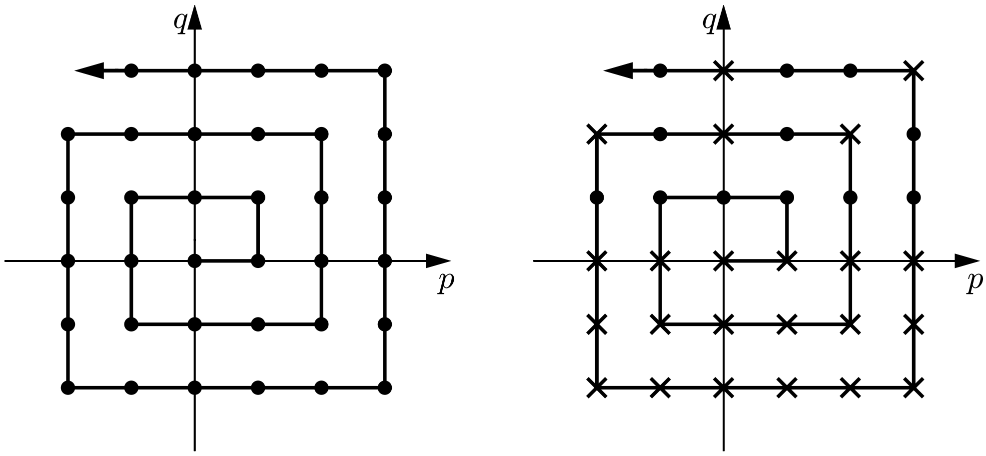

Figure A.1: Spiralling around we visit all points of $\mathbb{Z} \times \mathbb{Z}$ exactly once. Skipping some points and considering $\frac{p}{q}$ we visit all points of $\mathbb{Q}$ exactly once.

once, i.e., we have a sequence $(p_n, q_n) \in \mathbb{Z}\times\mathbb{Z}$ such that $\mathbb{Z}\times\mathbb{Z} = \{(p_n, q_n) \mid n \in \mathbb{N}$ and $n \neq m \Rightarrow (p_n, q_n) \neq (p_m, q_m)$. By skipping all points with $q_n \leq 0$ and considering $\frac{p_n}{q_n}$ we get a surjective map $\mathbb{N} \to \mathbb{Q}$ and by skipping numbers we already have we get a bijective map, i.e., $\mathbb{Q}$ is countable, see Figure A.1 right. The first few rational numbers are $1, 0, -1, 2, \frac{1}{2}, -\frac{1}{2}, -2, 3, \frac{3}{2}, \frac{2}{3}, \frac{1}{3}, -\frac{1}{3}, \cdots$.

We have just seen that $\mathbb{Q}$ is countable and now we will show that $\mathbb{R}$ is uncountable.

Theorem A.16. *The real numbers are uncountable.*

Proof. Assume we have a bijective map $\mathbb{N} \to \mathbb{R} : n \to x_n$. We can find an interval $[a_1, b_1]$ with $a_1 < b_1$ such that $x_1 \notin [a_1, b_1]$, e.g. $a_1 = x_1 + 1$ and $b_1 = x_1 + 2$. We now recursively want to find nested intervals $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ such that $x_n \notin [a_n, b_n]$ and $b_n - a_n = \frac{b_1 - a_1}{3^n}$.

Assume we have found $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}] \subseteq \cdots \subseteq [a_1, b_1]$, such that $a_k \notin [a_k, b_k]$ and $b_k - a_k = \frac{b_1 - a_1}{3^k}$ for $k = 1, \ldots, n$. Now consider the points $c = \frac{b_n + 2a_n}{3}$ and $d = \frac{2b_n + a_n}{3}$. They divide the interval $[a_n, b_n]$ into three parts $[a_n, c]$, $[c, d]$, and $[d, b_n]$ with length $\frac{b_n - a_n}{3}$. The number x_{n+1} cannot be an element of all three intervals so we can pick one of them that does not contain x_{n+1}. Call it $[a_{n+1}, b_{n+1}]$. We now have that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$, that $x_{n+1} \notin [a_{n+1}, b_{n+1}]$, and that $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{3} = \frac{b_1 - a_1}{3^{n+1}}$ as required.

As $b_n - a_n = \frac{b_1 - a_1}{3^n} \to 0$ for $n \to \infty$. The nested interval theorem tells us that there exists a number $x \in \mathbb{R}$ such that $\bigcap_{n \in \mathbb{N}} [a_n, b_n] = \{x\}$. Now $x \in [a_n, b_n]$ and $x_n \notin [a_n, b_n]$ so $x \neq x_n$ for all $n \in \mathbb{N}$ and that contradicts that the map $\mathbb{N} \to \mathbb{R} : n \mapsto x_n$ is surjective. □

A.3 A construction of the real numbers

There are more than one way to construct the real numbers, i.e., an ordered field where every non empty set bounded from below has an infimum. But Theorem A.14 shows that they all yield the same result.

The starting point for the approach we present is Corollary A.13, where we saw that in the end we must have $x = \inf(]x, \infty[\cap \mathbb{Q})$, or equivalent $x = \sup(]-\infty, x] \cap \mathbb{Q})$, If $A \subseteq \mathbb{Q}$ is a subset of the rational numbers then the *complement* is $A^{\complement} = \mathbb{Q} \setminus A$. Observe that $(]x, \infty[\cap \mathbb{Q})^{\complement} =]-\infty, x] \cap \mathbb{Q}$. Halflines in $\mathbb{Q}$, like $A =]x, \infty[\cap \mathbb{Q}$, have the following properties

$$A \neq \emptyset\,, \tag{A.25}$$

$$A \text{ is bounded from below}\,, \tag{A.26}$$

$$\forall x \in A\, \forall y \in \mathbb{Q} : x \leq y \implies y \in A\,, \tag{A.27}$$

$$\forall x \in A\, \exists y \in A : y < x\,. \tag{A.28}$$

Remark A.17. Condition (A.28) says that A does not have a minimum. In particular, if $x \in \mathbb{Q}$ then $A = [x, \infty[\cap \mathbb{Q} \subseteq \mathbb{Q}$ does not satisfies the condition.

Lemma A.18. *If A satisfies Condition (A.27) and $a \in A^{\complement}$ then a is a lower bound for A.*

Proof. Assume $a \in A^{\complement}$ and there exist $x \in A$ such that $x < a$ then (A.27) says that $a \in A$, a contradiction. □

Conversely,

Lemma A.19. *If A satisfies Condition (A.28) and a is a lower bound for A then $a \in A^{\complement}$.*

Proof. Assume a is a lower bound for A then Condition (A.28) tells us that $a \notin A$. ☐

All in all we have

Lemma A.20. *If A satisfies Condition (A.27) and (A.28) then $A^{\complement}$ is the set of lower bounds for A.*

This inspires the following definition of the real numbers as a set.

Definition A.21. The set of real numbers is

$$\mathbb{R} = \{A \subset \mathbb{Q} \mid A \text{ satisfies Condition (A.25), (A.26), (A.27), and (A.28)}\}$$

We now have to equip this space with an addition, a multiplication, and an ordering such that all the axioms of an ordered field is satisfied. If $]x, \infty[$ and $]y, \infty[$ are half lines then we have $]x, \infty[\supset]y, \infty[$ if and only if $x \leq y$. So we define the ordering by

Definition A.22. Let $A, B \in \mathbb{R}$ we say that $A \leq B$ if $A \supseteq B$.

We need to show that this defines a total ordering on $\mathbb{R}$, i.e., that it is reflexive, antisymmetric, transitive, and total, see Definition 1.1.

Lemma A.23. *Definition A.22 defines a total ordering on $\mathbb{R}$.*

Proof. Left as Exercise A.2 and A.3. ☐

Recall that if $A, B \subseteq \mathbb{Q}$ are arbitrary sets then

$$A + B = \{x + y \in \mathbb{Q} \mid x \in A \wedge y \in B\}.$$

We will use this as the definition of addition in $\mathbb{R}$, but first we need

Lemma A.24. *If $A, B \in \mathbb{R}$ then $A + B \in \mathbb{R}$.*

Proof. Assume $A, B \in \mathbb{R}$. As $A, B \neq \emptyset$ there exists $(x, y) \in A \times B$ and then $x + y \in A + B$ so $A + B \neq \emptyset$. So (A.25) is satisfied.

As A and B are bounded from below we can find $a, b \in \mathbb{Q}$ such that $a \leq x$ for all $x \in A$ and $b \leq y$ for all $y \in B$ but then $a + b \leq x + y$ for all $(x, y) \in A \times B$, i.e., $A + B$ is bounded from below by $a + b$. So (A.26) is satisfied.

Suppose $(x, y) \in A \times B$, $z \in \mathbb{Q}$ and $x + y \leq z$. Then $x \leq z - y$ and as $A \in \mathbb{R}$ we have $z - y \in A$ and hence $z = (z - y) + y \in A + B$. So (A.27) is satisfied.

Suppose $c \notin A + B$. We have just shown that $A + B$ satisfies (A.27) so c is a lower bound for $A + B$. If $c = x + y$ where $(x, y) \in A \times B$ then x must be a lower bound for A and hence a minimum for A but that contradicts Condition (A.28). So $A + B$ satisfies Condition (A.28). ☐

Definition A.25. Addition $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$A + B = \{x + y \in \mathbb{Q} \mid x \in A \wedge y \in B\} .$$

Lemma A.24 says that addition is well defined, but we still need to show that it satisfies Condition 1, 2, 4, and 5 in [1, Definition 6.1] and that it is compatible with the ordering (Condition (1.5) in Definition 1.1).

Lemma A.26. *Addition satisfies the commutative law and the associative law,*

$$A + B = B + A , \qquad (A + B) + C = A + (B + C) .$$

Proof. Left as Exercise A.4 and A.5. ☐

Lemma A.27. *The set $O = \{x \in \mathbb{Q} \mid 0 < x\}$ is a neutral element for addition in $\mathbb{R}$.*

Proof. Left as Exercise A.6. ☐

Lemma A.28. *Let $A \in \mathbb{R}$ and $X = \{-x \mid x \in A\}$. Then*

$$-A = \begin{cases} (X \setminus \{\max X\})^{\complement} , & \textit{if } X \textit{ has a maximum} , \\ X^{\complement} , & \textit{otherwise} , \end{cases}$$

is an additive inverse for A.

Proof. Left as Exercise A.7. ☐

Lemma A.29. *Addition in $\mathbb{R}$ is compatible with the ordering.*

Proof. Left as Exercise A.8. ☐

Lemma A.30. *Let $A \in \mathbb{R}$ then $A \leq O \iff O \leq -A$*

Proof. Left as Exercise A.9. ☐

Multiplication is a bit harder to define. Observe that if we for subsets $A, B \subseteq \mathbb{Q}$ put

$$A \odot B = \{xy \in \mathbb{Q} \mid x \in A \wedge y \in b\} , \tag{A.29}$$

and have $a, b \in \mathbb{Q}_+$ then

$$]a, \infty[\odot]b, \infty[=]ab, \infty[, \qquad]-a, \infty[\odot]b, \infty[=]-\infty, \infty[.$$

Definition A.31. The *non negative real numbers* are

$$\mathbb{R}_0 = \{A \in \mathbb{R} \mid O \leq A\} . \tag{A.30}$$

The *positive real numbers* are

$$\mathbb{R}_+ = \{A \in \mathbb{R} \mid O < A\} . \tag{A.31}$$

The *negative real numbers* are

$$\mathbb{R}_- = \{A \in \mathbb{R} \mid A < O\} . \tag{A.32}$$

Lemma A.32. *We have the following characterisations*

$$\mathbb{R}_0 = \{A \in \mathbb{R} \mid A \textit{ is bounded from below by } 0\} = \{A \in \mathbb{R} \mid 0 \notin A\}\,, \tag{A.33}$$
$$\mathbb{R}_+ = \{A \in \mathbb{R} \mid A \textit{ is bounded from below by a positive number}\}\,, \tag{A.34}$$
$$\mathbb{R}_- = \{A \in \mathbb{R} \mid A \textit{ contains a negative number}\} = \{A \in \mathbb{R} \mid 0 \in A\}\,. \tag{A.35}$$

Proof. Left as Exercise A.10. □

Lemma A.33. *If $A, B \in \mathbb{R}_0$ then $A \odot B \in \mathbb{R}_0$*

Proof. We have $A, B \neq \emptyset$ so there exists $x \in A$ and $y \in B$ then $xy \in A \odot B$. So $A \odot B \neq \emptyset$ and (A.25) is satisfied.

If $x \in A$ and $y \in B$ then $x, y > 0$ hence $xy > 0$ and we see that $A \odot B$ is bounded from below by 0. So (A.26) is satisfied.

If $x \in A$, $y \in B$, $z \in \mathbb{Q}$, and $0 < xy \leq z$ then $1 \leq \frac{z}{xy}$. Hence $x \leq x\frac{z}{xy} = \frac{z}{y}$ so $\frac{z}{y} \in A$ and $z = \frac{z}{y}y \in A \odot B$. So (A.27) is satisfied.

If $x \in A$ and $y \in B$ then we can find $x' \in A$ such that $x' < x$. This implies that $x'y \in A \odot B$ satisfies $x'y < xy$. So (A.28) is satisfied.

Finally, 0 is a lower bound for both A and B. So if $x, y \in A \times B$ then $0 \leq x, y$ and hence $0 \leq xy$, i.e., 0 is lower bound for $A \odot B$ and $A \odot B \in \mathbb{R}_0$. □

Definition A.34. Multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$A \cdot B = \begin{cases} A \odot B\,, & \text{if } A, B \in \mathbb{R}_0 \times \mathbb{R}_0\,, \\ -(-A \odot B)\,, & \text{if } A, B \in \mathbb{R}_- \times \mathbb{R}_0\,, \\ -(A \odot -B)\,, & \text{if } A, B \in \mathbb{R}_0 \times \mathbb{R}_-\,, \\ -A \odot -B\,, & \text{if } A, B \in \mathbb{R}_- \times \mathbb{R}_-\,. \end{cases} \tag{A.36}$$

It follows from Lemma A.33 that this is well defined. The details are left as Exercise A.11.

We still need to show that it satisfies Condition 1, 2, 3, 4, and 6 in [1, Definition 6.1] and that it is compatible with the ordering (Condition (1.5) in Definition 1.1)

Lemma A.35. *Multiplication satisfies the commutative law and the associative law,*

$$A \cdot B = B \cdot A\,, \qquad (A \cdot B) \cdot C = A \cdot (B \cdot C)\,.$$

Proof. Left as Exercise A.12 and A.13. □

Lemma A.36. *The set $I = \{x \in \mathbb{Q} \mid 1 < x\}$ is a neutral element for multiplication in $\mathbb{R}$.*

Proof. Left as Exercise A.14. □

Lemma A.37. *Let $A \in \mathbb{R}$ if $A \neq O$ then*

$$A^{-1} = \begin{cases} \left\{\frac{1}{x} \middle| x \in A\right\}, & A \in \mathbb{R}_+, \\ -(-A)^{-1}, & A \in \mathbb{R}_-, \end{cases}$$

is a multiplicative inverse for A.

Proof. Left as Exercise A.15. □

Lemma A.38. *Multiplication in $\mathbb{R}$ is compatible with the ordering.*

Proof. Left as Exercise A.16. □

We now have that $(\mathbb{R}, +, \cdot, \leq)$ is an ordered field. All that is left is to prove that it satisfies the nested interval theorem or equivalently that every subset bounded from below has an infimum. This is surprisingly easy.

Lemma A.39. *Let $\mathcal{A} \subseteq \mathbb{R}$ be non empty and bounded from below. Then*

$$\inf \mathcal{A} = \bigcup_{A \in \mathcal{A}} A. \tag{A.37}$$

Proof. We need to show that $\bigcup_{A\in\mathcal{A}} A$ satisfies the Conditions (A.25) , (A.26), (A.27), and (A.28). That it is a lower bound for $\mathcal{A}$, i.e., that $\bigcup_{A\in\mathcal{A}} A \leq A$ for all $A \in \mathcal{A}$, and finally if $\bigcup_{A\in\mathcal{A}} A \leq B$ for some $B \in \mathbb{R}$ and B is a lower bound for $\mathcal{A}$ then $B = \bigcup_{A\in\mathcal{A}} A$. This is left as Exercise A.17, A.18 and A.19. □

This proves

Theorem A.40. *The real numbers $\mathbb{R}$ satisfies one and hence all three properties in Theorem A.11.*

Lemma A.41. *The map $f : \mathbb{Q} \to \mathbb{R}$ by $f(x) =]x, \infty[$ preserves addition, multiplication, and the ordering and $f(1)$ is the identity in $\mathbb{R}$. That is, it makes $\mathbb{Q}$ into a subfield of $\mathbb{R}$.*

Proof. That $f(1)$ is the identity is obvious. We need to show that $f(x + y) = f(x) + f(y)$, $f(xy) = f(x) \cdot f(y)$, and that $x \leq y \implies f(x) \leq f(y)$. This is left as Exercise A.20, A.21, and A.22. □

If we identify $\mathbb{Q}$ with its image in $\mathbb{R}$ then Lemma A.12 tells us that $\mathbb{Q}$ is dense in $\mathbb{R}$: Between any two distinct real numbers is a rational number. At this point we can stop thinking about the real numbers as halflines in $\mathbb{Q}$ and just think of them as introducing new irrational numbers so as to "plug all holes in $\mathbb{Q}$". Rational numbers have a decimal expansion that are periodically, e.g. $\frac{743}{333} = 2.23\,123\,123\cdots$ and now we introduce numbers with arbitrary decimal expansions. e.g. $\pi = 3.14159265\cdots$. By truncating the decimal expansion we obtain a rational approximation to a given real number and inside a computer that is normally all we have. So what is the point of all this? The point is that we now know that $\mathbb{R}$ has the nested interval property. That can only be proved rigorously if we have a precise definition of $\mathbb{R}$.

A.4 Exercises

Exercise A.1. Prove that the map $f : \mathbb{Z} \to \mathbb{F}$ in Lemma A.1 is injective. Hint: Use that f preserves the ordering.

Exercise A.2. Show that that Definition A.22, ($A \leq B \iff A \supseteq B$) defines an ordering on $\mathbb{R}$, i.e., it is

- Reflexive: $A \leq A$ for all $A \in \mathbb{R}$.
- Antisymmetric: $A \leq B \wedge B \leq A \Rightarrow A = B$ for all $A, B \in \mathbb{R}$.
- Transitive: $A \leq B \wedge B \leq C \Rightarrow A \leq C$ for all $A, B, C \in \mathbb{R}$.

Exercise A.3. Show the ordering on $\mathbb{R}$ is total, i.e., for all $A, B \in \mathbb{R}$ we have $A \leq B$ and/or $B \leq A$.

Exercise A.4. Show that addition in $\mathbb{R}$ ($A + B = \{x + y \mid x \in A \wedge y \in B\}$) is commutative, i.e., $A + B = B + A$ for all $A, B \in \mathbb{R}$

Exercise A.5. Show that addition in $\mathbb{R}$ is associative, i.e., $(A+B)+C = A+(B+C)$ for all $A, B, C \in \mathbb{R}$.

Exercise A.6. Show that $O = \{x \in \mathbb{Q} \mid 0 < x\}$ is a neutral element for addition in $\mathbb{R}$.

Exercise A.7. Show that if $A \in \mathbb{R}$ and $X = \{-x \mid x \in A\}$. Then

$$-A = \begin{cases} (X \setminus \max X)^{\complement}, & \text{if } X \text{ has a maximum}, \\ X^{\complement}, & \text{otherwise}, \end{cases}$$

is an additive inverse for A. You need to show that $-A \in \mathbb{R}$ and that $A+(-A) = O$.

Exercise A.8. Show that addition in $\mathbb{R}$ is compatible with the ordering, i.e., $A \leq B \Rightarrow A + C \leq B + C$ for all $A, B, C \in \mathbb{R}$.

Exercise A.9. Show that $A \leq O \iff O \leq -A$ for all $A \in \mathbb{R}$.

Exercise A.10. Show that

$$\begin{aligned} \mathbb{R}_0 &= \{A \in \mathbb{R} \mid A \text{ is bounded from below by } 0\} = \{A \in \mathbb{R} \mid 0 \notin A\}, \\ \mathbb{R}_+ &= \{A \in \mathbb{R} \mid A \text{ is bounded from below by a positive number}\}, \\ \mathbb{R}_- &= \{A \in \mathbb{R} \mid A \text{ contains a negative number}\} = \{A \in \mathbb{R} \mid 0 \in A\}. \end{aligned}$$

Exercise A.11. Using Lemma A.33 show that multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (Definition A.34) is well defined, i.e., that if $A, B \in \mathbb{R}$ then $A \cdot B \in \mathbb{R}$.

Exercise A.12. Show that multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is commutative, i.e., that $A \cdot B = B \cdot A$ for all $A, B \in \mathbb{R}$.

Exercise A.13. . Show that multiplication $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is associative, i.e., that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ for all $A, B, C \in \mathbb{R}$.

Exercise A.14. Show that $I = \{x \in \mathbb{Q} \mid 1 < x\}$ is a neutral element for multiplication in $\mathbb{R}$.

Exercise A.15. Show that if $A \in \mathbb{R}$ if $A \neq O$ then

$$A^{-1} = \begin{cases} \left\{\frac{1}{x} \middle| x \in A\right\}, & A \in \mathbb{R}_+ , \\ -(-A)^{-1}, & A \in \mathbb{R}_- , \end{cases}$$

is a multiplicative inverse for A.

Exercise A.16. Show that multiplication in $\mathbb{R}$ is compatible with the ordering, i.e., if $A \leq B$ and $O \leq C$ then $A \cdot C \leq B \cdot C$.

Exercise A.17. Show that if $\mathcal{A} \subset \mathbb{R}$ is non empty and bounded from below then $\bigcup_{A \in \mathcal{A}} A \in \mathbb{R}$.

Exercise A.18. Show that if $\mathcal{A} \subset \mathbb{R}$ is non empty and bounded from below then $\bigcup_{A \in \mathcal{A}} A$ is a lower bound for $\mathcal{A}$.

Exercise A.19. Let $\mathcal{A} \subset \mathbb{R}$ be non empty and bounded from below. Show that if $B \in \mathbb{R}$ is a lower bound for $\mathcal{A}$ and $\bigcup_{A \in \mathcal{A}} A \leq B$ then $B = \bigcup_{A \in \mathcal{A}} A$.

Exercise A.20. Let $f : \mathbb{Q} \to \mathbb{R}$ be defined by by $f(x) =]x, \infty[$. Show that $f(x+y) = f(x) + f(x)$ for all $x, y \in \mathbb{Q}$.

Exercise A.21. Let $f : \mathbb{Q} \to \mathbb{R}$ be defined by by $f(x) =]x, \infty[$. Show that $f(xy) = f(x) \cdot f(x)$ for all $x, y \in \mathbb{Q}$.

Exercise A.22. Let $f : \mathbb{Q} \to \mathbb{R}$ be defined by by $f(x) =]x, \infty[$. Show that $x \leq y \Rightarrow f(x) \leq f(y)$.

Appendix B

Properties of normed vector spaces

Let $\|\cdot\|_2$ be the usual euclidean norm on $\mathbb{R}^n$, i.e, $\|\boldsymbol{x}\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$.

Theorem B.1. *If $(\boldsymbol{x}_m)_{m\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^n$ then it has a convergent subsequence.*

Proof. We use induction on the dimension n. The case $n = 1$ is Lemma 1.13. So assume the lemma is true for some $n \in \mathbb{N}$ and let $((x_{m;1}, \ldots, x_{m;n+1}))_{m\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}^{n+1}$. Then $((x_{m;1}, \ldots, x_{m;n}))_{m\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^n$. By the induction hypothesis we have convergent subsequence $((x_{m_k;1}, \ldots, x_{m_k;n}))_{k\in\mathbb{N}}$. The sequence $(x_{m_k;n+1})_{k\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$ and by Lemma 1.13 (case $n = 1$) it has a convergent subsequence $(x_{m_{k_\ell};n+1})_{\ell\in\mathbb{N}}$. Now $((x_{m_{k_\ell};1}, \ldots, x_{m_{k_\ell};n+1}))_{\ell\in\mathbb{N}}$ is convergent in $\mathbb{R}^{n+1}$. □

Theorem B.2. *If F is a closed and bounded subset of $\mathbb{R}^n$ and $f : F \to \mathbb{R}$ is a continuous function then f attains its maximum and minimum. That is, there exist $\boldsymbol{a}, \boldsymbol{b} \in F$ such that $f(\boldsymbol{a}) \le f(\boldsymbol{x}) \le f(\boldsymbol{b})$ for all $\boldsymbol{x} \in F$.*

Proof. Let $c = \inf_{\boldsymbol{x}\in F} f(\boldsymbol{x})$ and choose a sequence $(\boldsymbol{x}_m)_{m\in\mathbb{N}}$ in F such that $f(\boldsymbol{x}_m) \to c$ for $m \to \infty$. As f is bounded the sequence is bounded and by Theorem B.1 it has a convergent subsequence $(\boldsymbol{x}_{n_k})_{k\in\mathbb{N}}$. Let $\boldsymbol{a} = \lim_{k\to\infty} \boldsymbol{x}_{n_k}$ as F is closed we have $\boldsymbol{a} \in F$ and as f is continuous $f(\boldsymbol{a}) = \lim_{k\to\infty} f(\boldsymbol{x}_{n_k}) = c$. Starting with $d = \sup_{\boldsymbol{x}\in F} f(\boldsymbol{x})$ a similar argument shows that there exists $\boldsymbol{b} \in F$ such that $f(\boldsymbol{b}) = d$. □

Lemma B.3. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. There exist $C \in \mathbb{R}$ such that $\|\boldsymbol{x}\| \le C\|\boldsymbol{x}\|_2$ for all $\boldsymbol{x} \in \mathbb{R}^n$.*

Proof. Let $\boldsymbol{e}_1, \ldots \boldsymbol{e}_n$ be the standard basis for $\mathbb{R}^n$. If $\boldsymbol{x} = (x_1, \ldots, x_n)$ then we have

$$\|\boldsymbol{x}\| = \left\|\sum_{i=1}^n x_i\boldsymbol{e}_i\right\| \le \sum_{i=1}^n \|x_i\boldsymbol{e}_i\| = \sum_{i=1}^n |x_i|\|\boldsymbol{e}_i\|$$

$$\le \max_{i=1,\ldots,n} \|\boldsymbol{e}_i\| \sum_{i=1}^n |x_i| \le (n \max_{i=1,\ldots,n} \|\boldsymbol{e}_i\|)\|\boldsymbol{x}\|_2 \,. \quad \square$$

Corollary B.4. *An arbitrary norm* $\|\cdot\|$ *on* $\mathbb{R}^n$ *is a continuous function* $\mathbb{R}^n \to \mathbb{R}$ *w.r.t. the usual euclidean norm.*

Proof. By Lemma B.3 we have $\|\boldsymbol{x}\| \leq C\|\boldsymbol{x}\|_2$ for some $C \in \mathbb{R}$. If $\boldsymbol{x}_n \to \boldsymbol{x}$ for $n \to \infty$ then we have $|\|\boldsymbol{x}_n\| - \|\boldsymbol{x}\|| \leq \|\boldsymbol{x} - \boldsymbol{x}_n\| \leq C\|\boldsymbol{x} - \boldsymbol{x}_n\|_2 \to 0$ for $n \to \infty$, i.e., $\|\boldsymbol{x}_n\| \to \|\boldsymbol{x}\|$ for $n \to \infty$. □

Lemma B.5. *Le* $S^{n-1} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\|_2 = 1\}$ *be the standard unit sphere in* $\mathbb{R}^n$. *Then*

$$\sup_{\boldsymbol{x} \in S^{n-1}} \|\boldsymbol{x}\| < \infty\,, \qquad \inf_{\boldsymbol{x} \in S^{n-1}} \|\boldsymbol{x}\| > 0\,.$$

Proof. As S^{n-1} is bounded and closed Theorem B.2 give us $\boldsymbol{a}, \boldsymbol{b} \in S^{n-1}$ such that $\|\boldsymbol{a}\| = \inf_{\boldsymbol{x} \in S^{n-1}} \|\boldsymbol{x}\|$ and $\|\boldsymbol{b}\| = \sup_{\boldsymbol{x} \in S^{n-1}} \|\boldsymbol{x}\|$. We have $\|\boldsymbol{b}\| \in \mathbb{R}$ and as $\|\boldsymbol{a}\|_2 = 1$ we have $\boldsymbol{a} \neq \boldsymbol{0}$ and hence $\|\boldsymbol{a}\| > 0$. □

All norms on $\mathbb{R}^n$ are equivalent:

Theorem B.6. *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^n$. *There exist* $c, C > 0$ *such that* $c\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\| \leq C\|\boldsymbol{x}\|_2$ *for all* $\boldsymbol{x} \in \mathbb{R}^n$.

Proof. The existence of C is Lemma B.3. Let S^{n-1} be the standard unit sphere in $\mathbb{R}^n$ and put $c = \inf_{\boldsymbol{x} \in S^{n-1}} \|\boldsymbol{x}\|$. By Lemma B.5 we have $c > 0$.

If $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{x} \neq \boldsymbol{0}$ then $\left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}\right\|_2 = 1$. Hence $\left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}\right\| > c$ and

$$\|\boldsymbol{x}\| = \left\|\|\boldsymbol{x}\|_2 \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}\right\| = \|\boldsymbol{x}\|_2 \left\|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2}\right\| \geq c\|\boldsymbol{x}\|_2\,.$$ □

Appendix C

The trigonometric functions

The trigonometric functions are defined as the x and y coordinates of a point on the unit circle, respectively, see Figure C.1 left. If we rotate the picture with the angle ϕ

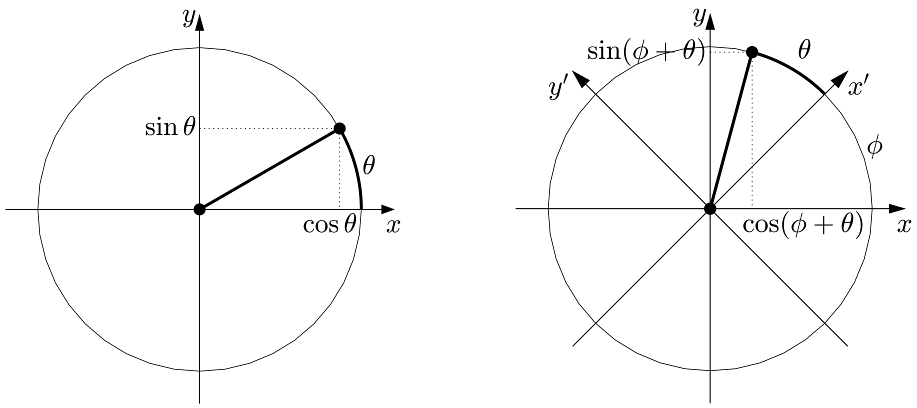

Figure C.1: The definition of cosine and sine.

then we obtain the picture in Figure C.1 right. On one hand the point $(\cos\theta, \sin\theta)$ is clearly moved to the point $(\cos(\theta+\phi), \sin(\theta+\phi))$. On the other hand, rotating with the angle ϕ is a linear map with the matrix $\left(\begin{smallmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{smallmatrix}\right)$. Applying this matrix to the point $(\cos\theta, \sin\theta)$ gives us

$$\begin{pmatrix}\cos(\theta+\phi)\\ \sin(\theta+\phi)\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \begin{pmatrix}\cos\phi\cos\theta - \sin\phi\sin\theta\\ \sin\phi\cos\theta + \cos\phi\sin\theta\end{pmatrix},$$

and we have derived the *addition identities* for cosine and sine.

We now want to show that cos and sin are differentiable in $\theta = 0$. In Figure C.2 we have

$$|AB| = \cos\theta\,, \qquad |BD| = \sin\theta\,, \qquad |CD| = \frac{|BD|}{\cos\theta} = \frac{\sin\theta}{\cos\theta}\,.$$

We have $0 < |BD| < |\theta| < |CD|$ for $|\theta| < \frac{\pi}{2}$ and $\frac{|BD|}{|CD|} = \cos\theta \to 1$ for $\theta \to 0$. This implies that $\frac{|\sin\theta|}{|\theta|} = \frac{|BD|}{|\theta|} \to 1$ for $\theta \to 0$. As $\sin 0 = 0$ this in turn proves that sin is differentiable at 0 with derivative 1.

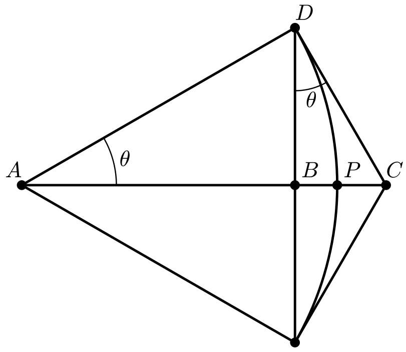

Figure C.2: The length of the circle segment PD is θ and $|BD| < |\theta| < |CD|$ if $|\theta| < \frac{\pi}{2}$.

We have $|BC| = |CD| \sin\theta = \frac{\sin^2\theta}{\cos\theta}$ so $\frac{|BC|}{\theta} = \frac{|\sin\theta|}{|\theta|}\frac{\sin\theta}{\cos\theta} \to 1 \cdot 0 = 0$ for $\theta \to 0$. As $|1-\cos\theta| < |BC|$ this implies that $\frac{|1-\cos\theta|}{|\theta|} \to 0$ for $\theta \to 0$. As $\cos 0 = 1$ this in turn proves that cos is differentiable at 0 with derivative 0.

For an arbitrary $\theta \in \mathbb{R}$ we now have

$$\begin{aligned} \cos(\theta + h) &= \cos\theta\cos h - \sin\theta\sin h\,, \\ \sin(\theta + h) &= \cos\theta\sin h + \sin\theta\cos h\,. \end{aligned}$$

We now have

$$\begin{aligned} \frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta} &= \left.\frac{\mathrm{d}\cos(\theta+h)}{\mathrm{d}h}\right|_{h=0} = \cos\theta \left.\frac{\mathrm{d}\cos h}{\mathrm{d}h}\right|_{h=0} - \sin\theta \left.\frac{\mathrm{d}\sin h}{\mathrm{d}h}\right|_{h=0} = -\sin\theta\,, \\ \frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta} &= \left.\frac{\mathrm{d}\sin(\theta+h)}{\mathrm{d}h}\right|_{h=0} = \cos\theta \left.\frac{\mathrm{d}\sin h}{\mathrm{d}h}\right|_{h=0} + \sin\theta \left.\frac{\mathrm{d}\cos h}{\mathrm{d}h}\right|_{h=0} = \cos\theta\,. \end{aligned}$$

Appendix D

The logarithm and exponential

One way of defining the *natural logarithm*, $\log : \mathbb{R}_+ \to \mathbb{R}$, is the following.

Definition D.1. For $x > 0$ we put

$$\log(x) = \int_1^x \frac{1}{t}\,\mathrm{d}t\,, \qquad x > 0\,. \tag{D.1}$$

The natural logarithm has the following basic properties:

Theorem D.2. *The natural logarithm,* log, *satisfies*

1. *It is differentiable with derivative* $x \mapsto \frac{1}{x}$.
2. *It is monotonically increasing, i.e.,* $0 < x < y \Rightarrow \log(x) < \log(y)$.
3. *The image is* $\mathbb{R}$*, i.e.,* $\log(\mathbb{R}_+) = \mathbb{R}$.
4. *If* $x, y > 0$ *then* $\log(xy) = \log(x) + \log(y)$.

Proof. 1. is the fundamental theorem of calculus (Theorem 2.61).

2. We integrate a positive function ($\frac{1}{x} > 0$ for $x > 0$). So if $0 < x < y$ then $\log(y) - \log(x) = \int_x^y \frac{1}{t}\,\mathrm{d}t > 0$, i.e., $\log(x) < \log(y)$.

For 4. we let $x, y > 0$. The substitution $t = xu$ shows that

$$\int_x^{xy} \frac{1}{t}\,\mathrm{d}t = \int_1^y \frac{1}{xu} x\,\mathrm{d}u = \int_1^y \frac{1}{u}\,\mathrm{d}u = \log(y)\,.$$

So

$$\log(xy) = \int_1^{xy} \frac{1}{t}\,\mathrm{d}t = \int_1^x \frac{1}{t}\,\mathrm{d}t + \int_x^{xy} \frac{1}{t}\,\mathrm{d}t = \log(x) + \log(y)\,.$$

Finally for 3. we note that for $n \in \mathbb{Z}$ Property 4 yields $\log(2^n) = n\log(2)$ and as $\log(2) > \log(1) = 0$ this implies that $\log(2^n) \to \pm\infty$ for $n \to \pm\infty$. So the image is all of $\mathbb{R}$. □

We see that $\log : \mathbb{R}_+ \to \mathbb{R}$ is bijective and hence has an inverse $\log^{-1} : \mathbb{R} \to \mathbb{R}_+$. We can defined the *exponential function* as this inverse:

Definition D.3. The exponential function $\exp : \mathbb{R} \to \mathbb{R}_+$ is $\exp = \log^{-1}$.

It has the following properties,

Theorem D.4. *The exponential function* $\exp : \mathbb{R} \to \mathbb{R}_+$ *satisfies*

1. *It is differentiable with derivative* exp.
2. *It is monotonically increasing.*
3. *The image is* $\mathbb{R}_+$.
4. *For* $x, y \in \mathbb{R}$ *we have* $\exp(x+y) = \exp(x)\exp(y)$.

Proof. The last three properties follows immediately from the corresponding properties of log. So we only need to consider the first. If $\log(x) = y$ then $\exp(y) = x$ and by Theorem 2.40 exp is differentiable and the derivative is

$$\exp'(y) = \frac{1}{\log'(x)} = \frac{1}{\left(\frac{1}{x}\right)} = x = \exp(y)\,. \qquad \square$$

As $\log(1) = 0$ we have $\exp(0) = 1$ and as $\exp(x)\exp(-x) = \exp(x-x) = \exp(0) = 1$ we have $\exp(-x) = \frac{1}{\exp(x)}$. If $n \in \mathbb{N}$ then $\exp(nx) = \exp(x)^n$ and $\exp(-nx) = \frac{1}{\exp(nx)} = \frac{1}{\exp(x)^n} = \exp(x)^{-n}$. We also have $\exp(x) = \exp\left(n\frac{1}{n}x\right) = \exp\left(\frac{1}{n}x\right)^n$ so $\exp\left(\frac{1}{n}x\right) = \sqrt[n]{\exp(x)} = \exp(x)^{\frac{1}{n}}$. Combining this we see that $\exp\left(\frac{p}{q}x\right) = \exp(x)^{\frac{p}{q}}$ for all $p \in \mathbb{Z}$ and all $q \in \mathbb{N}$.

Definition D.5. *Eulers constant* is the number $\exp(1)$ and is denoted e, i.e., $e = \exp(1)$.

As $\log(e) = \log(\exp(1)) = 1$ we can also define e by the condition $\int_1^e \frac{1}{t}\,\mathrm{d}t = 1$. For $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ we have $\exp\left(\frac{p}{q}\right) = \exp(1)^{\frac{p}{q}} = e^{\frac{p}{q}}$.

If $a > 0$, $p \in \mathbb{Z}$, and $q \in \mathbb{N}$ then $a^{\frac{p}{q}} = (\exp(\log(a)))^{\frac{p}{q}} = \exp\left(\frac{p}{q}\log(a)\right)$. Using the logarithm and the exponential we can define a^x for any power $x \in \mathbb{R}$:

Definition D.6. If $a > 0$ and $x \in \mathbb{R}$ then $a^x = \exp(x\log(a))$. In particular $e^x = \exp(x)$.

We see that $\log(a^x) = \log(\exp(x\log(a))) = x\log(a)$.

Definition D.7. If $a > 0$ then the *logarithm with base* a is defined as $\log_a(x) = \frac{\log(x)}{\log(a)}$. In particular $\log_e = \log$.

We see that $\log_a(a^x) = x$, i.e., the maps $x \mapsto a^x$ and $\log_a$ are each other inverses.

Exercises

Exercise D.1. Consider the proof of Theorem D.2. Why does $\log(2^n) \to \pm\infty$ for $n \to \pm\infty$ implies that the image of log is all of $\mathbb{R}$?

Exercise D.2. Prove the first three properties in Theorem D.4.

Exercise D.3. Let $x \in \mathbb{R}$. Use induction to prove that $\exp(xn) = \exp(x)^n$ for all $n \in \mathbb{N}$.

Exercise D.4. Let $x \in \mathbb{R}$. Prove that $\exp\left(\frac{p}{q}x\right) = \exp(x)^{\frac{p}{q}}$ for all $p \in \mathbb{Z}$ and all $q \in \mathbb{N}$.

Exercise D.5. Let $a > 0$ and $x \in \mathbb{R}$. Use that $a^x = \exp(x \log a)$ to find the derivative with respect to x, c.f. Example 2.19. Why is the function differentiable?

Exercise D.6. Let $a \in \mathbb{R}$ and $x > 0$. Use that $x^a = \exp(a \log x)$ to find the derivative with respect to x, c.f. Example 2.19. Why is the function differentiable?

Exercise D.7. Let $x > 0$. Use that $x^x = \exp(x \log x)$ to find the derivative with respect to x. Why is the function differentiable?

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