Danmarks Tekniske Universitet

Written exam, May 13, 2024
Course name: Mathematics 1b
Course number: 01004
Aids: All aids allowed by DTU (without internet)
Duration: 4 hours
Weights: Ex. 1: 20%, Ex. 2: 20%, Ex. 3: 20%, Ex. 4: 20%, Ex. 5: 20%.

In order to obtain full credit, you are required to provide complete arguments. The answers can be given in English or Danish. All references (terminology, definitions, etc.) are to the lecture notes. A Danish version of the exam set follows *after* the English version.

Exercise 1. Consider the quadratic form $q : \mathbb{R}^3 \to \mathbb{R}$ given by

$$q(\boldsymbol{x}) = 5x_1^2 + 5x_2^2 + 8x_3^2 + 8x_1x_2 - 4x_1x_3 + 4x_2x_3 - 22x_1 - 32x_2 - 20x_3 + 53,$$

where $\boldsymbol{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3$.

(a) Compute the gradient $\nabla q(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^3$.

(b) Compute the Hessian matrix $\boldsymbol{H}_q$. *Hint: The Hessian matrix should not depend on $\boldsymbol{x}$.*

(c) Find an orthonormal basis of eigenvectors for the Hessian matrix $\boldsymbol{H}_q$.

(d) Show that $(1, 2, 1)$ is a stationary point of q. Find all stationary points of q.

(e) State a direction from the stationary point $(1, 2, 1)$ in which the function q neither increases nor decreases (i.e., a direction where the function is constant).

(f) We now consider the gradient method with learning rate $\alpha = 0.02$ and initial guess $\boldsymbol{x}_0$ given by:

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

Compute $\boldsymbol{x}_{10}$, where

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \alpha \nabla q(\boldsymbol{x}_n), \quad n = 0.1, 2, \ldots.$$

You should state $\boldsymbol{x}_{10}$ as a vector of *decimal numbers* with an appropriate number of decimals. Which point does the gradient method converge to? (you do not have to provide a proof, just a qualified guess based on your computations).

The set of problems CONTINUES.

Exercise 2. Consider the quadratic form $q : \mathbb{R}^4 \to \mathbb{R}$ given by

$$q(\boldsymbol{x}) = 2x_1x_3 + 4x_2x_4,$$

where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T \in \mathbb{R}^4$.

(a) State a *symmetric* matrix $A \in \mathbb{R}^{4\times4}$ such that $q(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$, where $\boldsymbol{x} = [x_1, x_2, x_3, x_4]^T$.

(b) Find an orthogonal (change-of-basis) matrix $Q \in \mathbb{R}^{4\times4}$ that "reduces" the quadratic form q in the sense that q, in the new coordinates, does not contain "mixed terms" of the form x_ix_j (where $i \neq j$). Express q in the new coordinates.

We now consider q restricted to the set

$$B = \{\boldsymbol{x} \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1\}.$$

(c) Explain why the function $q : B \to \mathbb{R}$ has a minimal and maximal value.

(d) Determine the minimum value and the maximum value of $q : B \to \mathbb{R}$.

Exercise 3. Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{y^2\cos(x)}{x^2+y^2} & \text{for } (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}, \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

(a) Plot the graph of the function f.

(b) Compute the two first-order partial derivatives of f for $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

(c) Find the degree-two Taylor polynomial $P_2(x)$ of $\cos(x)$ at $x_0 = 0$.

(d) Show that

$$\lim_{x\to0} f(x, x) = \frac{1}{2}.$$

Hint: Use (c) and Taylor's limit formula.

(e) Determine the limit

$$\lim_{x\to0} f(x, 2x).$$

(f) Argue that f is differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$, but not in $(0, 0)$.

The set of problems CONTINUES.

Exercise 4. Consider the vector field $\boldsymbol{V} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\boldsymbol{V}(x, y, z) = (-x, xy^2, x + z)$$

and the curve $\mathcal{K}_1$ given by the parametrization:

$$\boldsymbol{r}(u) = (u, u^2, u + 1), \quad u \in [0, 2].$$

Thus, $\mathcal{K}_1 = \operatorname{im}(\boldsymbol{r})$.

(a) Find the tangent vector $\boldsymbol{r}'(u)$ and argue that the parameterization is regular.

(b) Calculate $\langle \boldsymbol{V}(\boldsymbol{r}(u)), \boldsymbol{r}'(u) \rangle$ for all $u \in [0, 2]$ and compute the line integral $\int_{\mathcal{K}_1} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$.

(c) Find a parametrization $\boldsymbol{p} : [0, 1] \to \mathbb{R}^3$ of the straight line from $(0, 0, 1)$ to $(2, 4, 3)$. We denote the line segment by $\mathcal{K}_2 = \operatorname{im}(\boldsymbol{p})$. Compute the line integral $\int_{\mathcal{K}_2} \boldsymbol{V} \cdot \mathrm{d}\boldsymbol{s}$.

(d) Determine if $\boldsymbol{V}$ is a gradient field.

Exercise 5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 + 1$$

and the subset $A \subset \mathbb{R}^2$ given by:

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid -2 \leq x_1 \leq 2 \ \wedge\ -1 \leq x_2 \leq 1\}.$$

(a) Compute the integral $\int_A f(x_1, x_2)\, \mathrm{d}(x_1, x_2)$.

(b) Determine the volume of the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in A \wedge 0 \leq x_3 \leq f(x_1, x_2)\}.$$

Let $a > 0$. Let $B \subset \mathbb{R}^2$ denote the circular disc centered at the origin with a radius of a:

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq a^2\}.$$

(c) Specify a parametrization of the circular disc B and find the associated Jacobian determinant. *Hint: Polar coordinates.*

(d) Determine the value of a to 3 decimal places such that

$$\int_A f(x_1, x_2)\, \mathrm{d}(x_1, x_2) = \int_B f(x_1, x_2)\, \mathrm{d}(x_1, x_2).$$

The set of problems is completed.